

Twisted hierarchies associated with the generalized sine-Gordon equation

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January 18, 2013

Abstract

Twisted U - and twisted U/K -hierarchies are soliton hierarchies introduced by Terng to find higher flows of the generalized sine-Gordon equation. Twisted $\frac{O(J,J)}{O(J) \times O(J)}$ -hierarchies are among the most important classes of twisted hierarchies. In this paper, interesting first and higher flows of twisted $\frac{O(J,J)}{O(J) \times O(J)}$ -hierarchies are explicitly derived, the associated submanifold geometry is investigated and a unified treatment of the inverse scattering theory is provided.

1 Introduction

The interaction between differential geometry and partial differential equations has been studied since the 19-th century and it can be found in the works of Lie, Darboux, Goursat, Bianchi, Bäcklund, and E. Cartan. One of the best known examples is the correspondence between surfaces of constant negative Gaussian curvature and the solutions of the sine-Gordon equation. The generalized sine-Gordon equation (GSGE) [25], [26]

$$\alpha \in O(n), \quad (1.1)$$

$$\partial_{x_j} \alpha_{ki} = \alpha_{kj} f_{ij}, \quad f_{ii} = 0, \quad i \neq j, \quad (1.2)$$

$$\partial_{x_j} f_{ij} + \partial_{x_i} f_{ji} + \sum_{k \neq i, j} f_{ik} f_{jk} = \alpha_{1i} \alpha_{1j}, \quad i \neq j \quad (1.3)$$

$$\partial_{x_k} f_{ij} = f_{ik} f_{kj}, \quad i, j, k \text{ distinct} \quad (1.4)$$

where $1 \leq i, j, k \leq n$, defined by the Gauss-Codazzi equation for n -dimensional submanifolds in \mathbb{R}^{2n-1} with constant sectional curvature -1 , is a natural multidimensional differential geometric generalization of the sine-Gordon equation.

The GSGE has Bäcklund transformations, permutability formula [26], [28], a Lax pair, and an inverse scattering theory [1]. Recently, via a Lie algebra splitting approach, Terng introduced twisted U - and twisted U/K -hierarchies (twisted hierarchies on symmetric spaces) [27] and showed that the GSGE can be interpreted as the 1-dimensional system of the twisted $\frac{O(n,n)}{O(n) \times O(n)}$ -hierarchy. As a result, she obtained all higher commuting flows of the GSGE.

There are rich intertwined analytic, algebraic and geometric structures of these twisted hierarchies. Twisted $\frac{O(J,J)}{O(J) \times O(J)}$ -hierarchies are among the most important classes of twisted hierarchies which contains the twisted $\frac{O(n,n)}{O(n) \times O(n)}$ -hierarchy ($J = I$) as a special case and possesses prototypical analytic, algebraic and geometric structures of twisted hierarchies. For instance, the associated Lax pairs of the twisted $\frac{O(J,J)}{O(J) \times O(J)}$ -hierarchy are Laurent polynomials in the spectral variable λ . Therefore, the eigenfunctions depend on x as $|\lambda| \rightarrow \infty$ and we need to renormalize the eigenfunctions in solving the inverse scattering problem. The renormalization process breaks the symmetries. Thus a proper gauge is needed to reconstruct the symmetries and the potentials. Inspired by the result of [1], we reconstruct the symmetry via solving an exterior product partial differential system which is derived from the associated 1-dimensional system.

On the other hand, in the study of the submanifold geometry associated with twisted hierarchies, besides the case $J = I$ mentioned above [26], [25], [1], we discover that the 1-dimensional systems of the twisted $\frac{O(J,J)}{O(J) \times O(J)}$ -hierarchy, $J = I_{1,n-1}$, describe the geometry of n -dimensional time-like submanifolds of constant positive sectional curvature in \mathbb{R}_1^{2n-1} . Therefore, rather than the space-like submanifolds associated with the generating equation introduced by Tenenblat [10], [24], the time-like submanifolds interpret the geometry of soliton equations and can be tackled via an inverse scattering method. In this respect, twisted $\frac{O(J,J)}{O(J) \times O(J)}$ -flows are more intimate with the GSGE or the sine-Gordon equation.

Finally, many interesting soliton hierarchies can be constructed from splittings of loop algebras fixed by involutions or automorphisms [2], [12], [29], [28]. This observation motivates the classification theory of integrable systems via different representations and possible reductions [20], [13], [21], [22], [19]. The set up of the correspondence between the reduction groups and the inverse scattering theory then becomes an important issue for mathematicians [16], [15], [17]. Since the twisted $\frac{O(J,J)}{O(J) \times O(J)}$ -hierarchies are integrable equations induced by two involutions

$$\sigma_0(\xi(-\lambda)) = \xi(\lambda), \quad \sigma_1(\xi(1/\lambda)) = \xi(\lambda),$$

on the loop group in the symmetric space $\frac{O(J,J)}{O(J) \times O(J)}$. Our work provides a complete inverse scattering theory of integrable systems with the reduction group given by the dihedral group D_2 .

The paper is organized as follows: in Section 2, we define the twisted $\frac{O(J,J)}{O(J) \times O(J)}$ -hierarchies via splittings of loop algebras and compute explicit examples which include a new 4-th order partial differential system (2.19), (2.20). Section 3 is devoted to the investigation of the associated submanifold geometry. In particular, we prove that the 1-dimensional twisted $\frac{O(J,J)}{O(J) \times O(J)}$ -system (twisted by σ_1) with $J = I_{1,n-1}$ is the Gauss-Codazzi equation for an n -dimensional time-like submanifold of constant sectional curvature 1 in \mathbb{R}_1^{2n-1} and derive a Bäcklund transformation theory for the 1-dimensional twisted $\frac{O(J,J)}{O(J) \times O(J)}$ -system. In Section 4, we solve the direct problem by constructing special eigenfunctions which corre-

spond to global twisted flows with nice decaying properties and regularities and extracting the scattering data. Section 5 and 6 are devoted to the reconstruction of the flows from scattering data. In particular, by studying the Riemann-Hilbert problem of the twisted $\frac{O(J,J)}{O(J) \times O(J)}$ -flows, eigenfunctions with arbitrary poles and multiplicity are constructed for $J = I$, and eigenfunctions with small purely continuous scattering data are derived for $J \neq I$. The Cauchy problems of twisted flows and the 1-dimensional twisted $\frac{O(J,J)}{O(J) \times O(J)}$ -system are solved in Section 7.

2 The twisted $\frac{O(J,J)}{O(J) \times O(J)}$ -hierarchy

To define the twisted $\frac{O(J,J)}{O(J) \times O(J)}$ -hierarchy via a loop group approach, for an integer q , $0 \leq q \leq n$, let us denote

$$J = I_{q,n-q} = \text{diag}(\overbrace{-1, \dots, -1}^{q \text{ times}}, \overbrace{1, \dots, 1}^{n-q \text{ times}}), \quad \tilde{J} = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}, \quad (2.1)$$

and

$$\begin{aligned} O(J, J) &= \left\{ x \in GL_{2n}(\mathbb{R}) \mid x^t \tilde{J} x = \tilde{J} \right\}, \\ o(J, J) &= \left\{ \xi \in gl_{2n}(\mathbb{R}) \mid \xi^t \tilde{J} + \tilde{J} \xi = 0 \right\}. \end{aligned}$$

For $i = 0, 1$, let σ_i be the involutions on $O(J, J)$ defined by

$$\sigma_i(x) = I_{n+i, n-i} x I_{n+i, n-i}^{-1},$$

and

$$o(J, J) = \mathcal{K}_i + \mathcal{P}_i$$

the Cartan decompositions for σ_i . So $K_0 \cap K_1 = S_0 \times S_1$, $K_0 = S_0 \times K'_0$, $K_1 = K'_1 \times S_1$ as direct product of subgroups with \mathcal{K}_i , \mathcal{S}_i , \mathcal{K}'_i be the Lie algebras of K_i , S_i , K'_i . More precisely,

$$\begin{aligned} \mathcal{K}_0 &= o(q, n-q) + o(q, n-q), & \mathcal{K}_1 &= \begin{cases} o(n, 1) + o(n-1), & \text{if } q = 0, \\ o(q, n-q+1) + o(q-1, n-q), & \text{if } q > 0, \end{cases} \\ \mathcal{S}_0 &= o(q, n-q) + 0_n, & \mathcal{S}_1 &= \begin{cases} 0_{n+1} + o(n-1), & \text{if } q = 0, \\ 0_{n+1} + o(q-1, n-q), & \text{if } q > 0, \end{cases} \\ \mathcal{K}'_0 &= 0_n + o(q, n-q), & \mathcal{K}'_1 &= \begin{cases} o(n, 1) + 0_{n-1}, & \text{if } q = 0, \\ o(q, n-q+1) + 0_{n-1}, & \text{if } q > 0. \end{cases} \end{aligned}$$

Let

$$\begin{aligned} L &= \{f : \mathbb{A}_{\epsilon, 1/\epsilon} \xrightarrow{\text{holo.}} GL_{2n}(\mathbb{C}) \mid (f(\bar{\lambda}))^* \tilde{J} f(\lambda) = \tilde{J}, \overline{f(\bar{\lambda})} = f(\lambda)\} \\ L^{\sigma_0} &= \{f \in L \mid \sigma_0(f(-\lambda)) = f(\lambda)\}, \\ L^{\sigma_0}_+ &= \{f \in L^{\sigma_0} \mid \sigma_1(f(1/\lambda)) = f(\lambda), f(1) \in K'_1\}, \\ L^{\sigma_0}_- &= \{f \in L^{\sigma_0} \mid f : \mathbb{C}/\mathbb{D}_\epsilon \xrightarrow{\text{holo.}} GL_{2n}(\mathbb{C}), f(\infty) \in K'_0\}. \end{aligned}$$

Here $\mathbb{S}^\epsilon, \mathbb{S}^{1/\epsilon}$ are circles of radius ϵ , and $1/\epsilon \neq 1$ centered at 0, \mathbb{D}_ϵ is the disk of radius $\epsilon \neq 1$, and $\mathbb{A}_{\epsilon, 1/\epsilon}$ is the annulus with boundaries \mathbb{S}^ϵ and $\mathbb{S}^{1/\epsilon}$. Then $L_+^{\sigma_0} \cap L_-^{\sigma_0} = \{1\}$ and the Lie algebras of $L^{\sigma_0}, L_+^{\sigma_0}, L_-^{\sigma_0}$ are

$$\begin{aligned}\mathcal{L}^{\sigma_0} &= \{\xi(\lambda) = \sum_{j \leq n_0} \xi_j \lambda^j \mid \xi_j \in \mathcal{K}_0 \text{ if } j \text{ is even, } \xi_j \in \mathcal{P}_0 \text{ if } j \text{ is odd}\}, \\ \mathcal{L}_+^{\sigma_0} &= \{\xi \in \mathcal{L}^{\sigma_0} \mid \xi_{-j} = \sigma_1(\xi_j), \xi(1) \in \mathcal{K}'_1\}, \\ \mathcal{L}_-^{\sigma_0} &= \{\xi \in \mathcal{L}^{\sigma_0} \mid \xi(\lambda) = \sum_{j \leq 0} \xi_j \lambda^j, \xi_0 \in \mathcal{K}'_0\}.\end{aligned}$$

It is a theorem that $(\mathcal{L}_+^{\sigma_0}, \mathcal{L}_-^{\sigma_0})$ is a splitting of \mathcal{L}^{σ_0} with $\hat{\pi}_\pm$ defined by

$$\hat{\pi}_+(\xi) = \pi_{\mathcal{S}_0}(\xi_0) - \pi_{\mathcal{S}_1} \left(\sum_{0 < j, j \text{ even}} (\xi_j + \sigma_1(\xi_j)) \right) + \sum_{0 < j \leq n_0} (\xi_j \lambda^j + \sigma_1(\xi_j) \lambda^{-j}) \quad (2.2)$$

$$\hat{\pi}_-(\xi) = \pi_{\mathcal{K}'_0}(\xi_0) + \pi_{\mathcal{S}_1} \left(\sum_{0 < j, j \text{ even}} (\xi_j + \sigma_1(\xi_j)) \right) + \sum_{0 < j \leq n_0} (\xi_{-j} - \sigma_1(\xi_j)) \lambda^{-j} \quad (2.3)$$

as the projections of $\xi = \sum_{j \leq n_0} \xi_j \lambda^j \in \mathcal{L}^{\sigma_0}$ onto $\mathcal{L}_\pm^{\sigma_0}$ with respect to the splitting [27]. Finally, let \mathcal{A} be a maximal abelian subalgebra of \mathcal{P}_0 consisting of elements of the form $\begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}$, where D is a diagonal matrix in $gl(n, \mathbb{R})$. Note $\sigma_1(\mathcal{A}) \subset \mathcal{A}$. Define

$$J_{a, 2j+1} = a \lambda^{2j+1} + \sigma_1(a) \lambda^{-(2j+1)} \in \mathcal{L}_+^{\sigma_0} \quad (2.4)$$

for some constant regular $a \in \mathcal{A}$.

Definition 2.1. The $2j+1$ -th twisted $\frac{O(J, J)}{O(J) \times O(J)}$ -flow (twisted by σ_1) is the compatibility condition of

$$[\partial_x + \hat{\pi}_+(M J_{a,1} M^{-1}), \partial_t + \hat{\pi}_+(M J_{\tilde{a}, 2j+1} M^{-1})] = 0, \quad (2.5)$$

for some $M = M(x, \lambda) \in L_-^{\sigma_0}$.

Theorem 2.1. Suppose $a = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}$, $\tilde{a} = \begin{pmatrix} 0 & \tilde{D} \\ \tilde{D} & 0 \end{pmatrix}$, $D = \text{diag}(w_1, \dots, w_n)$, $\tilde{D} = \text{diag}(\tilde{w}_1, \dots, \tilde{w}_n)$, $w_i \neq \pm w_j$, $\tilde{w}_i \neq \pm \tilde{w}_j$, for $i \neq j$. Then $2j+1$ -th twisted $\frac{O(J, J)}{O(J) \times O(J)}$ -flow is a nonlinear $2j+2$ -th order partial differential system in the components of b, v , with

$$\hat{\pi}_+(M J_{a,1} M^{-1}) = b a b^{-1} \lambda + v + \sigma_1(b a b^{-1}) \frac{1}{\lambda}, \quad b \in K'_0, v \in \mathcal{S}_0. \quad (2.6)$$

The proof of the above theorem follows from Lemma 2.1-2.3.

Lemma 2.1. The loop M in (2.5) can be chosen to satisfy

$$\begin{aligned}M(\partial_x + J_{a,1})M^{-1} &= \partial_x + \hat{\pi}_+(M J_{a,1} M^{-1}), \\ M(\partial_t + J_{\tilde{a}, 2j+1})M^{-1} &= \partial_t + \hat{\pi}_+(M J_{\tilde{a}, 2j+1} M^{-1}).\end{aligned}$$

Proof. Let $\Psi(x, t, \lambda)$ satisfy

$$\begin{aligned}\partial_x \Psi &= -\hat{\pi}_+ (M J_{a,1} M^{-1}) \Psi, \\ \partial_t \Psi &= -\hat{\pi}_+ (M J_{\tilde{a},2j+1} M^{-1}) \Psi.\end{aligned}$$

Write $\Psi(x, t, \lambda) = m(x, t, \lambda) e^{-x(\lambda a + \frac{1}{\lambda} \sigma_1(a)) - t(\lambda^{2j+1} \tilde{a} + \frac{1}{\lambda^{2j+1}} \sigma(\tilde{a}))}$. Then we derive

$$m(\partial_x + J_{a,1})m^{-1} = \partial_x + \hat{\pi}_+ (M J_{a,1} M^{-1}), \quad (2.7)$$

$$m(\partial_t + J_{\tilde{a},2j+1})m^{-1} = \partial_t + \hat{\pi}_+ (M J_{\tilde{a},2j+1} M^{-1}). \quad (2.8)$$

Hence

$$\hat{\pi}_+ (m J_{a,1} m^{-1}) = \hat{\pi}_+ (M J_{a,1} M^{-1}), \quad (2.9)$$

$$\hat{\pi}_+ (m J_{\tilde{a},2j+1} m^{-1}) = \hat{\pi}_+ (M J_{\tilde{a},2j+1} M^{-1}), \quad (2.10)$$

by taking the projection $\hat{\pi}_+$ on both sides of (2.7), (2.8). Plugging (2.9), (2.10) into (2.7), (2.8), we then have

$$\begin{aligned}m(\partial_x + J_{a,1})m^{-1} &= \partial_x + \hat{\pi}_+ (m J_{a,1} m^{-1}), \\ m(\partial_t + J_{\tilde{a},2j+1})m^{-1} &= \partial_t + \hat{\pi}_+ (m J_{\tilde{a},2j+1} m^{-1}).\end{aligned}$$

The property $m(x, t, \lambda) \in L_-^{\sigma_0}$ will be shown in Theorem 4.1. \square

Define the λ -coefficients of $\hat{\pi}_+(M J_{\tilde{a},2j+1} M^{-1})$, $\hat{\pi}_-(M J_{\tilde{a},2j+1} M^{-1})$ by

$$\hat{\pi}_+(M J_{\tilde{a},2j+1} M^{-1}) = \sum_{s=1}^{2j+1} Q_s(x) \lambda^s + Q_0(x) + \sum_{s=1}^{2j+1} \sigma_1(Q_s) \lambda^{-s}, \quad (2.11)$$

$$\hat{\pi}_-(M J_{\tilde{a},2j+1} M^{-1}) = R_0(x) + \sum_{s>0} R_s(x) \lambda^{-s} \quad (2.12)$$

by (2.2), (2.3).

Lemma 2.2. Let I be the $n \times n$ identity matrix, $U = \frac{1}{\sqrt{2}} \begin{pmatrix} I & -I \\ I & I \end{pmatrix}$, and

$$\begin{aligned}q_i(x) &= (bU)^{-1} Q_i bU, \quad 0 \leq i \leq 2j+1, \\ r_0(x) &= (bU)^{-1} R_0 bU, \\ f(x, \lambda) &= (bU)^{-1} M J_{\tilde{a},2j+1} M^{-1} bU \\ &= \sum_{s=1}^{2j+1} q_s \lambda^s + q_0 + r_0 + (bU)^{-1} \left(\sum_{s=1}^{2j+1} \sigma_1(Q_s) \lambda^{-s} + \sum_{s>0} R_s \lambda^{-s} \right) bU.\end{aligned}$$

Then

$$\begin{aligned}q_{2j+1} &= U^{-1} \tilde{a} U, \\ [U^{-1} a U, q_{2j}] &= -(bU)^{-1} (\partial_x Q_{2j+1}) bU - [(bU)^{-1} v bU, q_{2j+1}], \quad (2.13)\end{aligned}$$

$$\begin{aligned}[U^{-1} a U, q_s] &= -(bU)^{-1} (\partial_x Q_{s+1}) bU - [(bU)^{-1} v bU, q_{s+1}] \\ &\quad - [(bU)^{-1} \sigma_1(b a b^{-1}) bU, q_{s+2}], \quad 1 \leq s \leq 2j-1, s \neq 2j, 0\end{aligned} \quad (2.14)$$

$$\begin{aligned}[U^{-1} a U, q_0 + r_0] &= -(bU)^{-1} (\partial_x Q_1) bU - [(bU)^{-1} v bU, q_1] \\ &\quad - [(bU)^{-1} \sigma_1(b a b^{-1}) bU, q_2].\end{aligned} \quad (2.15)$$

Moreover,

$$\begin{aligned} & (f - \tilde{w}_1(\lambda^{2j+1} + \frac{1}{\lambda^{2j+1}})) \prod_{s=2}^n \left(f - \tilde{w}_s(\lambda^{2j+1} - \frac{1}{\lambda^{2j+1}}) \right) \\ & \cdot (f + \tilde{w}_1(\lambda^{2j+1} + \frac{1}{\lambda^{2j+1}})) \prod_{s=2}^n \left(f + \tilde{w}_s(\lambda^{2j+1} - \frac{1}{\lambda^{2j+1}}) \right) = 0. \end{aligned} \quad (2.16)$$

Proof. The identity (2.16) follows from the characteristic polynomial of f . By Lemma 2.1, we obtain

$$[\partial_x + \hat{\pi}_+ (MJ_{a,1}M^{-1}), MJ_{\tilde{a},2j+1}M^{-1}] = 0,$$

Therefore by (2.6), and (2.11), we derive

$$\begin{aligned} Q_{2j+1} &= b\tilde{a}b^{-1}, \\ [bab^{-1}, Q_{2j}] &= -\partial_x Q_{2j+1} - [v, Q_{2j+1}], \\ [bab^{-1}, Q_s] &= -\partial_x Q_{s+1} - [v, Q_{s+1}] - [\sigma_1(bab^{-1}), Q_{s+2}], \quad 1 \leq s \leq 2j-1, \\ [bab^{-1}, Q_0 + R_0] &= -\partial_x Q_1 - [v, Q_1] - [\sigma_1(bab^{-1}), Q_2]. \end{aligned}$$

Hence follows the lemma. \square

Lemma 2.3. For $\forall 0 \leq s \leq 2j$, the entries of Q_s are fixed functions of components of $\partial_x^\alpha b$ and $\partial_x^\beta v$, $0 \leq \alpha, \beta \leq 2j - s + 1$.

Proof. First of all, write $q_s = T_s + P_s$, for $s > 0$ and $q_0 + r_0 = T_0 + P_0$ with T_i, P_i being diagonal and off-diagonal respectively. Note that $U^{-1}aU$ is a diagonal matrix. Hence P_{2j} can be derived in terms of b, b_x, v by (2.13) once $w_i \neq \pm w_j$. Equating the $\lambda^{(2j+1)(2n)-1}$ -coefficients of the diagonal part of (2.16), we conclude $T_{2j} = 0$. Hence the lemma is done if $j = 0$. As a result, we can assume $j > 0$ in the following proof.

We are going to prove the lemma for $0 \leq s \leq 2j - 1$ by induction. Similarly, P_s can be derived in terms of $\partial_x^\alpha b, \partial_x^\beta v$, $0 \leq \alpha, \beta \leq 2j - s + 1$, by (2.14), (2.15) and the induction hypothesis. Using (2.16) and equating the $\lambda^{(2j+1)(2n)-(2j+1-s)}$ -coefficients of the diagonal part of (2.16), we obtain

$$T_s \text{diag} \left(\prod_{1 \leq k \leq 2n, k \neq 1} (\lambda_1 - \lambda_k), \dots, \prod_{1 \leq k \leq 2n, k \neq 2n} (\lambda_{2n} - \lambda_k) \right) = F_s(T_\alpha, P_\beta),$$

Here $\text{diag}(\lambda_1, \lambda_2, \dots, \lambda_{2n}) = \text{diag}(\tilde{w}_1, \dots, \tilde{w}_n, -\tilde{w}_1, \dots, -\tilde{w}_n)$, the entries of F_s are fixed polynomial functions of those of T_α, P_β , and $s+1 \leq \alpha \leq 2j+1, s \leq \beta \leq 2j+1$. Therefore, the lemma is justified if $\tilde{w}_i \neq \pm \tilde{w}_j$. \square

Via the algorithm provided in the proof of Theorem 2.1 and the Maple 7 software, we derive

Example 2.1. For $(n, q) = (2, 0)$, $a = \tilde{a}$ ($w_1 = \tilde{w}_1 = 1, w_2 = \tilde{w}_2 = 2$), *the first flow is the trivial linear system*

$$\partial_t u = \partial_x u, \quad \partial_t \omega = \partial_x \omega,$$

where

$$b = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos u(x, t) & \sin u(x, t) \\ 0 & 0 & -\sin u(x, t) & \cos u(x, t) \end{pmatrix}, \quad (2.17)$$

$$v = \begin{pmatrix} 0 & -\omega(x, t) & 0 & 0 \\ \omega(x, t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (2.18)$$

and **the third flow** of the twisted $\frac{O(J,J)}{O(J) \times O(J)}$ -hierarchy (twisted by σ_1) is the 4th order partial differential system

$$\begin{aligned} \partial_t u &= \frac{1}{18} \{ 10 \partial_x^3 u + \partial_x u [5(\partial_x u)^2 - 12\omega \partial_x u + 180(\cos u)^2 - 90 + 15\omega^2] \\ &\quad - 8\partial_x^2 \omega - 4\omega^3 + \omega [24 - 48(\cos u)^2] \}, \end{aligned} \quad (2.19)$$

$$\begin{aligned} \partial_t \omega &= -\frac{4}{9} \partial_x^4 u + \partial_x^2 u \left[-\frac{2}{3} (\partial_x u)^2 + \frac{5}{3} \omega \partial_x u - \frac{40}{3} (\cos u)^2 + \frac{20}{3} - \frac{2}{3} \omega^2 \right] \\ &\quad - 32 \sin u (\cos u)^3 + 16 \cos u \sin u \\ &\quad + \partial_x u \left[\frac{40}{3} (\partial_x u) \cos u \sin u + \frac{5}{6} (\partial_x u) (\partial_x \omega) - \frac{4}{3} \omega \partial_x \omega \right] \\ &\quad + \frac{5}{9} \partial_x^3 \omega + \partial_x \omega \left[\frac{5}{6} \omega^2 - 5 + 10(\cos u)^2 \right] - \frac{8}{3} \omega^2 \cos u \sin u. \end{aligned} \quad (2.20)$$

Here the associated Lax pair (2.5) is

$$\left[\partial_x + bab^{-1} \lambda + v + \sigma_1(bab^{-1}) \frac{1}{\lambda}, \partial_t + \sum_{s=1}^3 Q_s \lambda^s + Q_0 + \sum_{s=1}^3 \sigma_1(Q_s) \lambda^{-s} \right] = 0,$$

with b, v defined by (2.17), (2.18),

$$Q_3 = \begin{pmatrix} 0 & 0 & \cos u & -\sin u \\ 0 & 0 & 2 \sin u & 2 \cos u \\ \cos u & 2 \sin u & 0 & 0 \\ -\sin u & 2 \cos u & 0 & 0 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 0 & -\omega & 0 & 0 \\ \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial_x u \\ 0 & 0 & -\partial_x u & 0 \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 0 & 0 & \alpha_{11} & \alpha_{12} \\ 0 & 0 & \alpha_{21} & \alpha_{22} \\ \alpha_{11} & \alpha_{21} & 0 & 0 \\ \alpha_{12} & \alpha_{22} & 0 & 0 \end{pmatrix},$$

$$Q_0 = \begin{pmatrix} 0 & -\beta & 0 & 0 \\ \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\begin{aligned}
\alpha_{11} &= \frac{1}{3} (\sin u) \partial_x^2 u + \partial_x u \left[-\frac{1}{6} (\cos u) \partial_x u + \frac{2}{3} \omega \cos u \right] \\
&\quad - 2 (\cos u)^3 + 2 \cos u - \frac{1}{6} \omega^2 \cos u - \frac{2}{3} (\sin u) \partial_x \omega, \\
\alpha_{12} &= \frac{1}{3} (\cos u) \partial_x^2 u + \partial_x u \left[\frac{1}{6} (\sin u) \partial_x u - \frac{2}{3} \omega \sin u \right] \\
&\quad + \frac{1}{6} \omega^2 \sin u + 2 (\cos u)^2 \sin u - \frac{2}{3} (\cos u) \partial_x \omega, \\
\alpha_{21} &= \frac{2}{3} (\cos u) \partial_x^2 u + \partial_x u \left[\frac{1}{3} (\sin u) \partial_x u - \frac{1}{3} (\sin u) \omega \right] \\
&\quad + \frac{1}{3} \omega^2 \sin u + 4 (\cos u)^2 \sin u - \frac{1}{3} (\cos u) \partial_x \omega, \\
\alpha_{22} &= -\frac{2}{3} (\sin u) \partial_x^2 u + \partial_x u \left[\frac{1}{3} (\cos u) \partial_x u - \frac{1}{3} \omega \cos u \right] \\
&\quad + 4 (\cos u)^3 - 4 \cos u + \frac{1}{3} \omega^2 \cos u + \frac{1}{3} (\sin u) \partial_x \omega, \\
\beta &= -\frac{4}{9} \partial_x^3 u + \partial_x u \left[-\frac{2}{9} (\partial_x u)^2 + \frac{5}{6} \omega \partial_x u - 8 (\cos u)^2 + 4 - \frac{2}{3} \omega^2 \right] \\
&\quad + \frac{5}{9} \partial_x^2 \omega - \frac{5}{3} \omega + \frac{5}{18} \omega^3 + \frac{10}{3} \omega (\cos u)^2.
\end{aligned}$$

Example 2.2. If $(n, q) = (2, 0)$, and define a, \tilde{a} by $w_1 = 1, w_2 = 2, \tilde{w}_1 = 2, \tilde{w}_2 = 1, b, v'$ by (2.17), (2.18), then the first flow is the sine-Gordon equation

$$\partial_t^2 u - \partial_x^2 u = 12 \sin u \cos u, \quad \partial_t u = \omega.$$

Remark 1. For $i \in \{0, 1, \dots, n-1\}$, replacing σ_1 by σ_i , $\sigma_i(x) = I_{n+i, n-i} x I_{n+i, n-i}^{-1}$, and $\sigma_1(f(1/\lambda))$ either by $\sigma_i(f(1/\lambda))$ or by $\sigma_i(f(-1/\lambda))$, we can generalize the definition of twisted hierarchies by analogy.

3 The 1-dimensional system

We discuss associated submanifold geometry of twisted $\frac{O(J, J)}{O(J) \times O(J)}$ -flows. A 1-dimensional system is constructed by putting all first flows together in a soliton hierarchy. Many 1-dimensional systems are the Gauss-Codazzi equations for submanifolds in space forms or symmetric spaces with special geometric properties. For instance, the Gauss-Codazzi equations for isothermic surfaces in \mathbb{R}^3 is the 1-dimensional system of the $\frac{O(4, 1)}{O(3) \times O(1, 1)}$ -hierarchy [11], [6], [7]. Other interesting examples can be found in [27]. Similarly, for the 1-dimensional twisted $\frac{O(J, J)}{O(J) \times O(J)}$ -system, one has:

Definition 3.1. The 1-dimensional twisted $\frac{O(J, J)}{O(J) \times O(J)}$ -system (twisted by σ_1) is the compatibility condition of

$$[\partial_{x_i} + \hat{\pi}_+ (M J_{a_i, 1} M^{-1}), \partial_{x_j} + \hat{\pi}_+ (M J_{a_j, 1} M^{-1})] = 0, \quad 1 \leq i, j \leq n \quad (3.1)$$

for some $M = M(x_1, \dots, x_n, \lambda) \in L_-^{\sigma_0}$, and

$$a_i = \begin{pmatrix} 0 & e_i \\ e_i & 0 \end{pmatrix}, \quad e_i = \text{diag}(0, \dots, 0, \overset{i\text{-th entry}}{1}, 0, \dots, 0) \in \mathfrak{gl}(n, \mathbb{C}). \quad (3.2)$$

Example 3.1. (The sine-Gordon equation) The 1-dimensional twisted $\frac{O(J,J)}{O(J) \times O(J)}$ -system (twisted by σ_1) with $(n, q) = (2, 0)$ is the sine-Gordon equation.

Proof. In this case, we have $j = 0$, J in (2.1) is the 2×2 identity matrix, and

$$\begin{aligned} O(J, J) &= O(2, 2), & o(J, J) &= o(2, 2), \\ K_0 &= O(2) \times O(2), & K_1 &= O(2, 1) \times 1_1, \\ \mathcal{S}_0 &= o(2) + 0_2, & K'_0 &= 1_2 \times O(2). \end{aligned}$$

Let

$$a = a_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{a} = a_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad (3.3)$$

in the Lax pair (2.5) and write

$$v_1 = \begin{pmatrix} 0 & -\alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 & -\beta & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{S}_0, \quad (3.4)$$

$$b = \begin{pmatrix} 1 & 0 \\ 0 & \cos \frac{u}{2} & \sin \frac{u}{2} \\ & -\sin \frac{u}{2} & \cos \frac{u}{2} \end{pmatrix} \in K'_0 \quad (3.5)$$

in (2.6) and $\hat{\pi}_+(MJ_{a_2,1}M^{-1}) = ba_2b^{-1}\lambda + v_2 + \sigma_1(ba_2b^{-1})\frac{1}{\lambda}$. Equating the λ -coefficients of (2.5), we then derive

$$\partial_t(ba_1b^{-1}) - \partial_x(ba_2b^{-1}) + [v_2, ba_1b^{-1}] - [v_1, ba_2b^{-1}] = 0, \quad (3.6)$$

$$\partial_tv_1 - \partial_xv_2 - [v_1, v_2] - [\sigma_1(ba_1b^{-1}), ba_2b^{-1}] + [\sigma_1(ba_2b^{-1}), ba_1b^{-1}] = 0. \quad (3.7)$$

Plugging (3.4), (3.5) into (3.6), (3.7), we obtain

$$\alpha = \frac{1}{2}\partial_t u, \quad \beta = \frac{1}{2}\partial_x u, \quad \partial_x \beta - \partial_t \alpha = 2 \sin u.$$

Hence the 1-dimensional twisted $\frac{O(2,2)}{O(2) \times O(2)}$ -system (twisted by σ_1) is the sine-Gordon equation

$$u_{xx} - u_{tt} = 4 \sin u. \quad (3.8)$$

□

Example 3.2. (The sinh-Gordon equation) The 1-dimensional twisted $\frac{O(J,J)}{O(J) \times O(J)}$ -system (twisted by σ_1) with $(n, q) = (2, 1)$ is the sinh-Gordon equation.

Proof. We have $j = 0$, $J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, and

$$\begin{aligned} K_0 &= O(1, 1) \times O(1, 1), & K_1 &= O(1, 2) \times 1_1, \\ \mathcal{S}_0 &= o(1, 1) \times 0_2, & K'_0 &= 1_2 \times O(1, 1). \end{aligned}$$

Let

$$\begin{aligned} a = a_1 &= \begin{pmatrix} & & 1 & 0 \\ 0 & & 0 & 0 \\ 1 & 0 & & \\ 0 & 0 & & 0 \end{pmatrix}, \quad \tilde{a} = a_2 = \begin{pmatrix} & & 0 & 0 \\ 0 & & 0 & 1 \\ 0 & 0 & & \\ 0 & 1 & & 0 \end{pmatrix} \\ v_1 &= \begin{pmatrix} 0 & \alpha & & \\ \alpha & 0 & 0 & \\ & 0 & 0 & \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 & \beta & & \\ \beta & 0 & 0 & \\ & 0 & 0 & \end{pmatrix} \in \mathcal{S}_0 = o(1, 1) \times 0, \\ b &= \begin{pmatrix} 1 & & 0 & \\ & \cosh \frac{u}{2} & \sinh \frac{u}{2} & \\ 0 & \sinh \frac{u}{2} & \cosh \frac{u}{2} & \end{pmatrix} \in K'_0 = 1 \times O(1, 1). \end{aligned}$$

So a similar argument yields

$$\alpha = -\frac{1}{2}\partial_t u, \quad \beta = -\frac{1}{2}\partial_x u, \quad \partial_x \beta - \partial_t \alpha = -2 \sinh u$$

and

$$u_{xx} - u_{tt} = 4 \sinh u. \quad (3.9)$$

□

Example 3.3. (*The generalized sine-Gordon equation*) The 1-dimensional twisted $\frac{O(J, J)}{O(J) \times O(J)}$ -system (twisted by σ_1) with $(n, q) = (n, 0)$ is the Gauss-Codazzi equation for an n -dimensional submanifold of constant sectional curvature -1 in \mathbb{R}^{2n-1} , i.e. the generalized sine-Gordon equation (GSGE).

Proof. We give an alternative proof (cf [27]). The Gauss-Codazzi equation for an n -dimensional submanifold in \mathbb{R}^{2n-1} of constant sectional curvature -1 is the generalized sine-Gordon equation (GSGE) (1.1)-(1.4). Moreover, the Bäcklund transformation for the GSGE is constructed by showing that

$$dX - X\omega = D_\lambda \alpha \delta - X \delta \alpha^t D_\lambda X \quad (3.10)$$

gives a new solution to the GSGE if α is a given solution of the GSGE. Here

$$\omega = \delta F - F^t \delta, \quad \delta = \sum_{j=1}^n e_j dx_j, \quad D_\lambda = \frac{1}{2}(\lambda I - \frac{1}{\lambda} I_{1, n-1})$$

$F = (f_{ij})$ is defined by (1.2), e_j is defined as in (3.2), and I is the $n \times n$ identity matrix [25], [26], [1]. Later the Bäcklund transformation (3.10) is linearized (so is the GSGE (1.1)-(1.4)) by the Lax pair

$$\partial_{x_j} \tilde{\Psi} = \left(\frac{1}{2} \lambda A_j + \frac{1}{2\lambda} B_j + C_j \right) \tilde{\Psi}, \quad (3.11)$$

with $A_j = \begin{pmatrix} 0 & \alpha e_j \\ (\alpha e_j)^t & 0 \end{pmatrix}$, $B_j = \begin{pmatrix} 0 & -I_{1,n-1} \alpha e_j \\ -(\alpha e_j)^t I_{1,n-1} & 0 \end{pmatrix}$, $C_j = \begin{pmatrix} 0 & 0 \\ 0 & \gamma_j \end{pmatrix}$, $\gamma_j = C(\frac{\partial}{\partial x_j}) \in o(n)$, and solved by the inverse scattering method [1]. By a permutation,

$$Y \in gl(2n, \mathbb{C}) \quad \mapsto \quad PYP^t, \\ P = \begin{pmatrix} 0 & \iota \\ I & 0 \end{pmatrix}, \quad \iota(e_{n-j}) = e_{j+1}, \quad 0 \leq j \leq n-1,$$

and a change of coordinates

$$\begin{aligned} x_{n-j} &\mapsto 2x_{j+1}, \quad 1 \leq j \leq n, \\ \lambda &\mapsto -\lambda, \end{aligned}$$

the Lax pair (3.11) can be written as

$$\partial_{x_j} \Psi = -(ba_j b^{-1} \lambda + v_j + \sigma_1(ba_j b^{-1}) \frac{1}{\lambda}) \Psi,$$

where a_j is defined by (3.2), and

$$\begin{aligned} b &= \begin{pmatrix} I & 0 \\ 0 & g \end{pmatrix} \in K'_0 = 1 \times O(n), \quad g = \alpha \iota, \\ v_j &= \begin{pmatrix} u_j & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{S}_0 = o(n) \times 0, \quad u_j = -2\iota \gamma_j \iota. \end{aligned}$$

□

Example 3.4. (The generalized sinh-Gordon equation) The 1-dimensional twisted $\frac{O(J,J)}{O(J) \times O(J)}$ -system (twisted by σ_1) with $(n, q) = (n, 1)$ is the Gauss-Codazzi equation for an n -dimensional time-like submanifold of constant sectional curvature 1 in \mathbb{R}_1^{2n-1} which possesses simultaneously diagonalizable first and second fundamental forms.

To prove the statement of Example 3.4, we first show:

Theorem 3.1. Suppose M is a time-like n -dimensional submanifold of constant sectional curvature 1 in \mathbb{R}_1^{2n-1} . Suppose that there exist local coordinates x_1, x_2, \dots, x_n on a neighborhood of $p \in M$, an $O(1, n-1)$ -valued map $A = (a_j^i)$, and parallel normal frames e_{n+1}, \dots, e_{2n-1} such that the first and second fundamental forms are of the form

$$I = \sum_{i=1}^n \epsilon_i (a_i^1)^2 dx_i \otimes dx_i, \quad II = \sum_{\lambda=2}^n \sum_{i=1}^n \epsilon_i a_i^1 a_i^\lambda dx_i \otimes dx_i e_{n+\lambda-1}, \quad (3.12)$$

where $\text{diag}(\epsilon_1, \epsilon_2, \dots, \epsilon_n) = \text{diag}(-1, 1, \dots, 1) = I_{1, n-1} = J$. Then the Gauss-Codazzi equation and the structure equation is the 1-dimensional twisted $\frac{O(J, J)}{O(J) \times O(J)}$ -system (3.1) with

$$\begin{aligned}\hat{\pi}_+(MJ_{a,i}M^{-1}) &= ba_i b^{-1} \lambda + v_i + \sigma_1(ba_i b^{-1}) \frac{1}{\lambda}, \\ g = b^{-1} &= \begin{pmatrix} I_n & 0 \\ 0 & A^t \end{pmatrix} : \mathbb{R}^n \rightarrow K'_0 = I \times O(1, n-1), \\ a_i &= \frac{1}{2} \begin{pmatrix} 0 & e_i \\ e_i & 0 \end{pmatrix}, \quad e_i \text{ defined by (3.2)} \\ v_i &= \begin{pmatrix} u_i & 0 \\ 0 & 0 \end{pmatrix} : \mathbb{R}^n \rightarrow \mathcal{S}_0 = o(1, n-1) \oplus 0, \quad 1 \leq i \leq n.\end{aligned}\tag{3.13}$$

Moreover, writing $A = (a_j^i)$, the 1-dimensional twisted $\frac{O(J, J)}{O(J) \times O(J)}$ -system can be written as

$$A \in O(1, n-1), \tag{3.14}$$

$$\partial_{x_j} a_i^k = a_j^k f_{ij}, \quad f_{ii} = 0, \quad i \neq j, \tag{3.15}$$

$$\epsilon_j \partial_{x_j} f_{ij} + \epsilon_i \partial_{x_i} f_{ji} + \sum_{k \neq i, j} \epsilon_k f_{ik} f_{jk} = -a_i^1 a_j^1, \quad i \neq j \tag{3.16}$$

$$\partial_{x_k} f_{ij} = f_{ik} f_{kj}, \quad i, j, k \text{ distinct} \tag{3.17}$$

where $1 \leq i, j, k \leq n$. We call the nonlinear system (3.14)-(3.17) the **generalized sinh-Gordon equation**.

Proof. Step 1: the Gauss-Codazzi equation

To write down the Gauss-Codazzi equations for these immersions we set

$$\begin{aligned}\omega^i &= a_i^1 dx_i, \\ \omega_i^{n+\lambda-1} &= \epsilon_i a_i^\lambda dx_i,\end{aligned}$$

where $1 \leq i, j \leq n, 2 \leq \lambda \leq n$. Hence by the structure equations

$$d\omega^i = \omega^j \wedge \omega_j^i, \quad \epsilon_i \omega_j^i + \epsilon_j \omega_i^j = 0,$$

$\omega_j^i = f_{ij} dx_i - \epsilon_i \epsilon_j f_{ji} dx_j$, where

$$f_{ij} = \begin{cases} \frac{(a_i^1)_{x_j}}{a_j^1}, & i \neq j; \\ 0, & i = j. \end{cases} \tag{3.18}$$

Set $F = (f_{ij})$ and $\delta = \sum_{j=1}^n e_j dx_j$. Then

$$\omega = (\omega_j^i)_{1 \leq i, j \leq n} = \delta F - J F^t \delta J$$

is the Levi-Civita $o(1, n-1)$ -connection of the induced pseudo-Riemannian metric I. The Gauss-Codazzi equation and the structure equation give

$$\begin{cases} d\omega + \omega \wedge \omega = \delta A^t e_1 \wedge (AJ\delta) = -\delta A^t e_1 \wedge (JAJ\delta), \\ (a_i^\lambda)_{x_j} = a_j^\lambda f_{ij}, \quad 1 \leq i, j \leq n, i \neq j, 2 \leq \lambda \leq n. \end{cases} \tag{3.19}$$

Moreover, it follows from (3.18), (3.19) that

$$(a_i^k)_{x_j} = a_j^k f_{ij}, \quad 1 \leq i, j, k \leq n, \quad i \neq j. \quad (3.20)$$

On the other hand, since $A = (a_j^i) \in O(1, n-1)$,

$$\sum_j \epsilon_j (a_j^k)^2 = \epsilon_k.$$

Taking differential with respect to x_i on the above equality, we get

$$\epsilon_i a_i^k (a_i^k)_{x_i} = - \sum_{j \neq i} \epsilon_j a_j^k (a_j^k)_{x_i}.$$

It follows from (3.20) that

$$(a_i^k)_{x_i} = -\epsilon_i \sum_{j \neq i} \epsilon_j a_j^k f_{ji}. \quad (3.21)$$

Then (3.20) and (3.21) can be expressed as

$$A^{-1}dA = \delta F^t - JF\delta J.$$

Summarize, $A = (a_j^i)$ satisfies the following second order PDE system:

$$\begin{cases} d\omega + \omega \wedge \omega = \delta A^t e_1 \wedge AJ\delta = -\delta A^t e_1 \wedge (JAJ\delta), \\ A^{-1}dA = \delta F^t - JF\delta J, \\ \text{where } \omega = \delta F - JF^t \delta J, \end{cases} \quad (3.22)$$

Step 2: the 1-dimensional twisted $\frac{O(J,J)}{O(J) \times O(J)}$ -system

Define

$$\theta_\lambda = \sum_{i=1}^n ((g^{-1}a_i g)\lambda + v_i + \sigma_1(g^{-1}a_i g)\lambda^{-1})dx_i \quad (3.23)$$

$$= \frac{\lambda}{2} \begin{pmatrix} 0 & \delta A^t \\ JAJ\delta & 0 \end{pmatrix} + \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} - \frac{\lambda^{-1}}{2} \begin{pmatrix} 0 & \delta A^t J \\ AJ\delta & 0 \end{pmatrix}, \quad (3.24)$$

where $u = \sum_{i=1}^n u_i dx_i$, $A \in O(1, n-1)$. So (3.23) implies that the 1-dimensional twisted $\frac{O(J,J)}{O(J) \times O(J)}$ -system is equivalent to the flatness condition

$$d\theta_\lambda + \theta_\lambda \wedge \theta_\lambda = 0. \quad (3.25)$$

On the other hand, by (3.24), the flatness condition (3.25) is equivalent to (A, u) satisfying the following system

$$\begin{cases} -\delta \wedge dA^t + u \wedge \delta A^t = 0 \Leftrightarrow \delta \wedge (JA^{-1}dAJ) + u \wedge \delta = 0, & (*) \\ (JA^{-1}dAJ) \wedge \delta + \delta \wedge u = 0 \Leftrightarrow (*), \\ du + u \wedge u + \delta A^t e_1 \wedge JAJ\delta = 0. \end{cases}$$

The first equation implies that there exists $H = (h_{ij})$ with $h_{ii} = 0$ for all $1 \leq i \leq n$ such that $u = \delta H - JH^t \delta J$, $JA^{-1}dAJ = J\delta H^t J - H\delta \Leftrightarrow A^{-1}dA = \delta H^t - JH\delta J$. Thus the 1-dimensional twisted $\frac{O(J,J)}{O(J) \times O(J)}$ -system is given by the following PDEs:

$$\begin{aligned} u &= \delta H - JH^t \delta J, \\ A^{-1}dA &= \delta H^t - JH\delta J, \\ du + u \wedge u + \delta A^t e_1 \wedge JAJ\delta &= 0. \end{aligned} \tag{3.26}$$

Comparing (3.22) with (3.26), the first assertion of Theorem 3.1 is proved by setting $H = F$, $u = \omega$.

Step 3: the generalized sinh-Gordon equation

Formula (3.15) is exactly (3.20). Taking the coefficients of $dx_i \wedge dx_k$ of the ij -entry of both sides of the first equation of (3.22), we obtain (3.17). Similarly, (3.16) can be derived by taking the coefficients of $dx_i \wedge dx_j$ of the ij -entry of both sides of the first equation of (3.22). \square

Proof. of Example 3.4 :

Theorem 3.1 reduces the proof to showing the existence of such n -dimensional submanifolds. Note Theorem 7.2 of Section 7 implies that the 1-dimensional twisted $\frac{O(J,J)}{O(J) \times O(J)}$ -system (3.1) with b, a_i, v_i defined by (3.13) can be solved in \mathbb{R}^n . By Theorem 3.1, we then conclude the solvability of the Gauss-Codazzi equation of such submanifolds. Therefore, a modified version of the Bonnet Theorem yields the existence of a time-like n -dimensional submanifold M of constant sectional curvature 1 in \mathbb{R}_1^{2n-1} , local coordinates x_1, x_2, \dots, x_n on a neighborhood of $p \in M$, and parallel normal frames e_{n+1}, \dots, e_{2n-1} with the first and second fundamental forms (3.12). \square

We remark that the correspondence between the sinh-Gordon equation and the positive constant Gaussian curvature time-like surface in \mathbb{R}_1^3 has been established by Chern [8]. In the following theorem, we construct a Riccati type Bäcklund transformation, analogous to (3.10), of the generalized sinh-Gordon equation (3.14)-(3.17). Moreover, we linearize the Bäcklund transformation.

Theorem 3.2. *Suppose $A \in O(1, n-1)$ is a solution of the generalized sinh-Gordon equation and λ is a non-zero **real** constant. Consider the linear system for $y : \mathbb{R}^n \rightarrow \mathcal{M}_{n \times 2n}$:*

$$dy = y \begin{pmatrix} \omega & \delta A^t D_\lambda \\ D_\lambda JAJ\delta & 0 \end{pmatrix}, \quad D_\lambda = \frac{1}{2}(\lambda I - \lambda^{-1}J). \tag{3.27}$$

Then

1. System (3.27) is solvable.
2. If $y = (P, Q)$ is a solution of (3.27) with $Q \in GL(n)$, then $X = -Q^{-1}P \in O(1, n-1)$ is a solution of the Bäcklund transformation for the generalized sinh-Gordon equation given by

$$dX = X\delta A^t D_\lambda X + X\omega - D_\lambda JAJ\delta. \tag{3.28}$$

and X is again a solution of the generalized sinh-Gordon equation.

Proof. Define θ_λ by

$$\theta_\lambda = \begin{pmatrix} \omega & \delta A^t D_\lambda \\ D_\lambda J A J \delta & 0 \end{pmatrix}.$$

The assumption that A is a solution of the generalized sinh-Gordon equation gives the flatness of $\theta(\lambda)$ for any $\lambda \in \mathbb{C}$, which can imply the solvability of (3.27).

To prove the second statement, let

$$d(P, Q) = (P, Q)\theta_\lambda$$

or equivalently,

$$\begin{cases} dP = P\omega + QD_\lambda J A J \delta, \\ dQ = P\delta A^t D_\lambda. \end{cases}$$

By a direct computation, (3.28) is satisfied.

On the other hand, the assumption of $A \in O(1, n-1)$ being a solution of the generalized sinh-Gordon equation implies (3.22). Hence

$$\begin{aligned} X^{-1}dX &= \delta A^t D_\lambda X + \omega - X^{-1}D_\lambda J A J \delta \\ &= \delta A^t D_\lambda X + (\delta F - JF^t \delta J) - JX^t D_\lambda A \delta J \\ &= \delta(A^t D_\lambda X + F) - J(X^t D_\lambda A + F^t) \delta J \\ &:= \delta \tilde{F}^t - J\tilde{F} \delta J, \end{aligned}$$

where $\tilde{F} = X^t D_\lambda A + F^t$. Moreover, let

$$\begin{aligned} \tilde{\omega} &:= \delta \tilde{F} - J\tilde{F}^t \delta J \\ &= \delta(X^t D_\lambda A + F^t) - J(A^t D_\lambda X + F) \delta J \\ &= \delta X^t D_\lambda A - J A^t D_\lambda X \delta J + A^{-1} dA. \end{aligned}$$

Then $(\tilde{\omega}, X)$ satisfy the following system:

$$\begin{cases} d\tilde{\omega} + \tilde{\omega} \wedge \tilde{\omega} + \delta X^t D_\lambda^2 \wedge (JX \delta J) = 0 \\ \tilde{\omega} = \delta \tilde{F} - J\tilde{F}^t \delta J \\ X^{-1}dX = \delta \tilde{F}^t - J\tilde{F} \delta J, \end{cases}$$

where $\tilde{F} = X^t D_\lambda A + F^t$. It is equivalent to

$$\tilde{\theta}_\lambda = \begin{pmatrix} \tilde{\omega} & \delta X^t D_\lambda \\ D_\lambda J X J \delta & 0 \end{pmatrix}$$

is flat. By the argument in the proof of Theorem 3.1, X is also a solution of the generalized sinh-Gordon equation. \square

Remark 2. (The generating equation) *The Gauss-Codazzi equation for an n -dimensional Riemannian submanifold of constant sectional curvature K with flat normal bundle in a*

($2n-1$)-dimensional Riemannian or pseudo-Riemannian manifold (of index s) of constant sectional curvature \overline{K} is **the generating equation**

$$\begin{aligned}\alpha &\in O(n-q, q), \\ \partial_{x_j} \alpha_{ki} &= \alpha_{kj} f_{ji}, \quad f_{ii} = 0, \quad i \neq j, \\ \partial_{x_i} f_{ij} + \partial_{x_j} f_{ji} + \sum_{k \neq i, j} f_{ki} f_{kj} &= -K \alpha_{1i} \alpha_{1j}, \quad i \neq j \\ \partial_{x_k} f_{ij} &= f_{ik} f_{kj}, \quad i, j, k \text{ distinct}\end{aligned}$$

where $1 \leq i, j, k \leq n$, $q = s$ if $K < \overline{K}$ and $q = n - (s+1)$ if $K > \overline{K}$ [10], [3], [24]. Moreover, the Bäcklund transformation for the generating equation is constructed by showing that

$$dX + X \hat{J}^{-1/2} C \hat{J}^{1/2} = \Lambda_\lambda \alpha \delta \hat{J}^{1/2} - X \hat{J}^{-1/2} \delta \alpha^t \Lambda_\lambda \hat{J} X \quad (3.29)$$

gives a new solution to the generating equation if α is a given solution of the generating equation. Here

$$\begin{aligned}\hat{J} &= \text{diag}(\overbrace{1, \dots, 1}^{n-q \text{ times}}, \overbrace{-1, \dots, -1}^{q \text{ times}}), \\ C &= F\delta - \delta F^t, \quad \delta = \sum_{j=1}^n e_j dx_j, \quad \Lambda_\lambda = \frac{1}{2}(\lambda I + \frac{K}{\lambda} I_{1, n-1})\end{aligned}$$

$F = (f_{ij})$, e_j is defined as in (3.2), and I is the $n \times n$ identity matrix [9], [10], [24]. Similarly, the Bäcklund transformation (3.29) is linearized by the Lax pair

$$\partial_{x_j} \tilde{\Psi} = \left(\frac{1}{2} \lambda A_j \mp \frac{1}{2\lambda} B_j + C_j \right) \tilde{\Psi}, \quad (3.30)$$

(the \mp sign corresponds to $K = \pm 1$) with

$$A_j = \begin{pmatrix} 0 & \alpha \hat{J}^{\frac{1}{2}} e_j \\ e_j \hat{J}^{-\frac{1}{2}} \alpha^t \hat{J} & 0 \end{pmatrix}, \quad (3.31)$$

$$B_j = \begin{pmatrix} 0 & -I_{1, n-1} \alpha \hat{J}^{\frac{1}{2}} e_j \\ -e_j \hat{J}^{-\frac{1}{2}} \alpha^t I_{1, n-1} \hat{J} & 0 \end{pmatrix}, \quad (3.32)$$

$$C_j = \begin{pmatrix} 0 & 0 \\ 0 & \hat{J}^{-\frac{1}{2}} \gamma_j \hat{J}^{\frac{1}{2}} \end{pmatrix}, \quad (3.33)$$

$\gamma_j \in o(n)$. However, it is impossible to transform (3.30)-(3.33) into a twisted $\frac{O(J, J)}{O(J) \times O(J)}$ -system (twisted by σ_1) because the reality conditions fail by observing $A_j, B_j, C_j \notin o(J, J)$ unless $\hat{J} = I$. In particular, let $n = 2$,

$$\begin{aligned}\hat{J} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \alpha &= \begin{pmatrix} \cosh \frac{u}{2} & \sinh \frac{u}{2} \\ \sinh \frac{u}{2} & \cosh \frac{u}{2} \end{pmatrix} \in O(1, 1), \\ \gamma_1 &= \begin{pmatrix} 0 & \alpha \\ -\alpha & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \in o(n)\end{aligned}$$

in (3.31)-(3.33), then the compatibility conditions of (3.30) are

$$\alpha = -\frac{1}{2}\partial_{x_2}u, \quad \beta = \frac{1}{2}\partial_{x_1}u, \quad \partial_{x_1}\beta - \partial_{x_2}\alpha = \mp\frac{1}{2}\sinh u.$$

Therefore we obtain the **sinh-Laplace equation**

$$u_{x_1x_1} + u_{x_2x_2} = \mp \sinh u. \quad (3.34)$$

Here the \mp sign corresponds to $K = \pm 1$. We remark that the correspondence between the sinh-Laplace equation and the negative constant Gaussian curvature space-like surface in \mathbb{R}_1^3 has been discovered by Hu [18].

4 The direct scattering problem

Using (2.6), the linear spectral problem corresponding to (2.5) is

$$\begin{aligned} \frac{\partial \Psi}{\partial x} &= -\lambda bab^{-1}\Psi - \frac{1}{\lambda}\sigma_1(bab^{-1})\Psi - v\Psi, \\ b &\in K'_0, \quad v \in \mathcal{S}_0. \end{aligned} \quad (4.1)$$

In this section, we center on the construction of special eigenfunctions $\Psi(x, \lambda)$. By the normalization

$$\begin{aligned} \Psi(x, \lambda) &= m(x, \lambda)e^{-x(\lambda a + \frac{1}{\lambda}\sigma_1(a))} \\ &= b\check{m}(x, \lambda)e^{-x(\lambda a + \frac{1}{\lambda}\sigma_1(a))}, \end{aligned} \quad (4.2)$$

the linear spectral problem (4.1) turns into

$$\frac{\partial m}{\partial x} = \lambda(ma - bab^{-1}m) + \frac{1}{\lambda}(m\sigma_1(a) - \sigma_1(bab^{-1})m) - vm, \quad (4.3)$$

$$\frac{\partial \check{m}}{\partial x} = [\check{m}(x, \lambda), \lambda a + \frac{1}{\lambda}\sigma_1(a)] + Q(x, \lambda)\check{m}(x, \lambda), \quad (4.4)$$

with

$$Q(x, \lambda) = \frac{1}{\lambda}(\sigma_1(a) - b^{-1}\sigma_1(bab^{-1})b) - (b^{-1}\frac{\partial b}{\partial x} + b^{-1}vb). \quad (4.5)$$

Definition 4.1. We define the operator \mathcal{J}_λ on $gl(n, \mathbb{C})$ by

$$\mathcal{J}_\lambda f = \left[f, \lambda a + \frac{1}{\lambda}\sigma_1(a) \right],$$

and $\pi_0^\lambda, \pi_\pm^\lambda$ to be the orthogonal projections of $gl(n, \mathbb{C})$ to the $0-, \pm$ -eigenspaces of $\text{Re } \mathcal{J}_\lambda = \frac{1}{2}(\mathcal{J}_\lambda + (\mathcal{J}_\lambda)^*)$. Moreover, the characteristic curve of (4.1) is defined by

$$\Sigma_a = \left\{ \lambda \in \mathbb{C} \mid \text{the image of } \pi_0^\lambda \text{ is non-empty} \right\}.$$

Definition 4.2. We call $a = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}$ a principal oblique direction if $|w_1| > |w_\nu|$ for $1 < \nu \leq n$, and the $2n$ real numbers $\{\pm w_1, \dots, \pm w_n\}$ are distinct with $D = \text{diag}(w_1, \dots, w_n)$ [1]. One can verify that if a is a principal oblique direction, then $\Sigma_a = i\mathbb{R} \cup \mathbb{S}^1$. Let us label the components of $\mathbb{C} \setminus \Sigma_a$ as

$$\begin{aligned} \Omega^+ &= \{|\lambda| > 1, \text{Re}(\lambda) > 0\}, \\ \Omega^- &= \{|\lambda| > 1, \text{Re}(\lambda) < 0\}, \\ D^+ &= \{|\lambda| < 1, \text{Re}(\lambda) < 0\}, \\ D^- &= \{|\lambda| < 1, \text{Re}(\lambda) > 0\}. \end{aligned}$$

Note permutaion matrices do not commute with σ_1 . Hence it is natural to consider the following example.

Definition 4.3. We call $a = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}$ an oblique direction if the $2n$ real numbers $\{\pm w_1, \dots, \pm w_n\}$ are distinct. Here $D = \text{diag}(w_1, \dots, w_n)$. One can verify that if a is an oblique direction, then $\Sigma_a = i\mathbb{R} \cup \mathbb{S}^1 \cup_{1 \leq \nu \leq s} (\mathbb{S}^{r_\nu} \cup \mathbb{S}^{1/r_\nu})$, where $r_\nu = r_\nu(w_1, w_\nu) \neq 1$, and s is the number of w_ν such that $|w_\nu| > |w_1|$. Let us label the components of $\mathbb{C} \setminus \Sigma_a$ as

$$\begin{aligned} \Omega_\nu^+ &= \{r_\nu < |\lambda| < r_{\nu+1}, \text{Re}(\lambda) > 0\}, \\ \Omega_\nu^- &= \{r_\nu < |\lambda| < r_{\nu+1}, \text{Re}(\lambda) < 0\}, \\ D_\nu^+ &= \{1/r_{\nu+1} < |\lambda| < 1/r_\nu, \text{Re}(\lambda) < 0\}, \\ D_\nu^- &= \{1/r_{\nu+1} < |\lambda| < 1/r_\nu, \text{Re}(\lambda) > 0\}, \end{aligned}$$

for $0 \leq \nu \leq s$. Here we assume $r_0 = 1 < r_1 < \dots < r_s < r_{s+1} = \infty$.

Restricted to the case of $q = 0$ and a is a principal oblique direction, the direct problem is solved by [1], [5] after a diagonalization process of (4.1).

Theorem 4.1. Let $a \in \mathcal{A}$, a constant oblique direction, $b(x) \in K'_0$, $v(x) \in \mathcal{S}_0$. If $|b - 1|_{L_1^1 \cap L_\infty} + |v|_{L_1} < \infty$, then there exists a bounded set $Z \subset \mathbb{C}$, such that $Z \cap (\mathbb{C} \setminus \Sigma_a)$ is discrete in $\mathbb{C} \setminus \Sigma_a$ and for $\forall \lambda \in \mathbb{C} \setminus \Sigma_a$, there exists uniquely a solution $m(x, \lambda)$ of (4.3) satisfying:

$$m(\cdot, \lambda) \text{ is bounded, for each } \lambda \in \mathbb{C} \setminus (\Sigma_a \cup Z), \quad (4.6)$$

$$m(x, \lambda) \rightarrow 1 \text{ as } x \rightarrow -\infty, \text{ for each } \lambda \in \mathbb{C} \setminus (\Sigma_a \cup Z), \quad (4.7)$$

$$m(x, \cdot) \text{ is meromorphic in } \mathbb{C} \setminus \Sigma_a \text{ with poles at } \lambda \in Z, \quad (4.8)$$

$$m(x, \lambda) \rightarrow b(x) \text{ uniformly as } \lambda \rightarrow \infty, \quad (4.9)$$

and

$$m(x, \lambda) \in L_-^{\sigma_0}, \quad (4.10)$$

$$m(x, \lambda) = \sigma_1(m(x, 1/\lambda)). \quad (4.11)$$

Proof. Step 1: (Small data problem)

In this step, we assume that

$$|Q|_{L_1} < 1, \quad \text{for } \forall |\lambda| \geq 1,$$

where Q is defined by (4.5). Thus for $(x, \lambda) \in \mathbb{R} \times \Omega_\nu^\pm$ we can find $m'(x, \lambda)$ which satisfies the integral equation

$$\begin{aligned} m'(x, \lambda) = 1 &+ \int_{-\infty}^x e^{(x-y)\mathcal{J}_\lambda} (\pi_0^\lambda + \pi_-^\lambda) (Q(y, \lambda) m'(y, \lambda)) dy \\ &- \int_x^\infty e^{(x-y)\mathcal{J}_\lambda} \pi_+^\lambda (Q(y, \lambda) m'(y, \lambda)) dy. \end{aligned}$$

One can also verify that

$m'(x, \lambda)$ satisfies (4.4), (4.6), (4.10) for $(x, \lambda) \in \mathbb{R} \times \Omega_\nu^\pm$,
 $m'(x, \lambda) \rightarrow 1$ as $x \rightarrow -\infty$ or $|\lambda| \rightarrow \infty$,
 $m'(x, \lambda)$ is holomorphic in $\lambda \in \Omega_\nu^\pm$, and has a continuous extension to Σ_a from Ω_ν^\pm .

Define

$$m(x, \lambda) = \begin{cases} b(x)m'(x, \lambda), & \text{if } (x, \lambda) \in \mathbb{R} \times \Omega_\nu^\pm; \\ \sigma_1(b(x)m'(x, \frac{1}{\lambda})), & \text{if } (x, \lambda) \in \mathbb{R} \times D_\nu^\pm. \end{cases} \quad (4.12)$$

Using the σ_1 -symmetry and the unique solvability of (4.3), we then prove the theorem provided $|Q|_{L_1} < 1$, i.e. when the potentials (b, v) satisfy $|b - 1|_{L_1^1 \cap L_\infty} + |v|_{L_1} < c < 1$.

Step 2: (Large data problem)

We induce on the least integer $N \geq 0$ such that $|Q|_{L_1} < 2^N$. Note that the eigenfunction of (4.4) corresponding to a translate of Q is the translate (with respect to x) of the eigenfunction m' . Thus without loss of generality, we may assume that

$$Q_- = \begin{cases} Q, & x \leq 0, \\ 0, & x \geq 0, \end{cases} \quad Q = Q_+ + Q_-, \quad |Q_\pm|_{L_1} < 2^N.$$

The induction assumption implies that Q_- has an eigenfunction $\eta(x, \lambda)$, Q_+ has an eigenfunction $\rho(x, \lambda)$ (proved by analogy) satisfying

$$\eta(x, \lambda) \text{ satisfies (4.4), (4.6), (4.10) for } (x, \lambda) \in \mathbb{R}^- \times \Omega_\nu^\pm, \quad (4.13)$$

$$\rho(x, \lambda) \text{ satisfies (4.4), (4.6), (4.10) for } (x, \lambda) \in \mathbb{R}^+ \times \Omega_\nu^\pm, \quad (4.14)$$

$$\eta(x, \lambda) \rightarrow 1 \text{ as } x \rightarrow -\infty, \rho(x, \lambda) \rightarrow 1 \text{ as } x \rightarrow \infty, \quad (4.15)$$

$$\eta(x, \lambda), \rho(x, \lambda) \text{ are meromorphic in } \lambda \in \Omega_\nu^\pm \text{ and tend to 1 as } \lambda \rightarrow \infty. \quad (4.16)$$

Let us define

$$S^\pm(\lambda) = \rho^{-1}(0, \lambda) \eta(0, \lambda) \quad \text{for } \lambda \in \Omega_\nu^\pm. \quad (4.17)$$

One can adapt the argument in [14] to factorize

$$S^\pm(\lambda) = (1 + L^\pm(\lambda)) \delta^\pm(\lambda) (1 + U^\pm(\lambda))^{-1}, \quad (4.18)$$

where

$$\pi_0^\lambda(U^\pm(\lambda)) = \pi_-^\lambda(U^\pm(\lambda)) = 0, \quad (4.19)$$

$$\pi_0^\lambda(L^\pm(\lambda)) = \pi_+^\lambda(L^\pm(\lambda)) = 0, \quad (4.20)$$

$$\pi_+^\lambda(\delta^\pm(\lambda)) = \pi_-^\lambda(\delta^\pm(\lambda)) = 0, \quad (4.21)$$

for $\lambda \in \Omega_\nu^\pm$ and

$$U^\pm, L^\pm, \delta^\pm \text{ are meromorphic in } \lambda \in \Omega_\nu^\pm \text{ with poles at } Z, \quad (4.22)$$

$$U^\pm, L^\pm \text{ tend to } 0, \delta^\pm(\lambda) \text{ tends to } 1 \text{ uniformly as } |\lambda| \rightarrow \infty. \quad (4.23)$$

Where $Z = \{\text{zeros of minors of } P_{\pm,\nu}^{-1}S^\pm P_{\pm,\nu}\}$ and $P_{\pm,\nu}$ satisfies that

$$P_{\pm,\nu}^{-1}J_{a,1}P_{\pm,\nu} \text{ is diagonal,}$$

$$\text{real parts of the entries of } P_{\pm,\nu}^{-1}J_{a,1}P_{\pm,\nu} \text{ are nondecreasing}$$

on Ω_ν^\pm . Define

$$m'(x, \lambda) = \begin{cases} \eta(x, \lambda)e^{x\mathcal{J}_\lambda}(1 + U^\pm(\lambda)), & (x, \lambda) \in \{x \leq 0\} \times \Omega_\nu^\pm; \\ \rho(x, \lambda)(e^{x\mathcal{J}_\lambda}(1 + L^\pm(\lambda)))\delta^\pm(\lambda), & (x, \lambda) \in \{x \geq 0\} \times \Omega_\nu^\pm, \end{cases} \quad (4.24)$$

and $m(x, \lambda)$ by (4.12). Then we complete the theorem by properties (4.12)-(4.24). \square

Corollary 4.1. *Let $a \in \mathcal{A}$, a constant oblique direction, $b(x) \in K'_0$, $v(x) \in \mathcal{S}_0$, $|b - 1|_{L_1^1 \cap L_\infty} + |v|_{L_1} < \infty$. If the set Z in Theorem 4.1 is a finite set contained in $\mathbb{C} \setminus \Sigma_a$, then we have the following factorization properties for $m(x, \lambda)$:*

$$m(x, \lambda) = \begin{cases} b(x)\eta_\nu^\pm(x, \lambda)e^{x\mathcal{J}_\lambda}(1 + U_\nu^\pm(\lambda)), & (x, \lambda) \in \{x \leq 0\} \times \Omega_\nu^\pm, \\ b(x)\rho_\nu^\pm(x, \lambda)e^{x\mathcal{J}_\lambda}(1 + L_\nu^\pm(\lambda))\delta_\nu^\pm(\lambda), & (x, \lambda) \in \{x \geq 0\} \times \Omega_\nu^\pm; \\ b(x)\tilde{\eta}_\nu^\pm(x, \lambda)e^{x\mathcal{J}_\lambda}(1 + \tilde{U}_\nu^\pm(\lambda)), & (x, \lambda) \in \{x \leq 0\} \times D_\nu^\pm, \\ b(x)\tilde{\rho}_\nu^\pm(x, \lambda)e^{x\mathcal{J}_\lambda}(1 + \tilde{L}_\nu^\pm(\lambda))\tilde{\delta}_\nu^\pm(\lambda), & (x, \lambda) \in \{x \geq 0\} \times D_\nu^\pm, \end{cases} \quad (4.25)$$

with

$\eta_\nu^\pm(x, \lambda)$, $\rho_\nu^\pm(x, \lambda)$ satisfy (4.4), are uniformly bounded, and tend to 1 as $x \rightarrow \mp\infty$ respectively,

$\eta_\nu^\pm(x, \lambda)$, and $\rho_\nu^\pm(x, \lambda)$ are holomorphic and tend to 1 as $|\lambda| \rightarrow \infty$, $\delta_\nu^\pm(\lambda)$ are meromorphic and tend to 1 as $|\lambda| \rightarrow \infty$,

$$\pi_0^\lambda(U_\nu^\pm(\lambda)) = \pi_-^\lambda(U_\nu^\pm(\lambda)) = 0,$$

$$\pi_0^\lambda(L_\nu^\pm(\lambda)) = \pi_+^\lambda(L_\nu^\pm(\lambda)) = 0,$$

$$\pi_+^\lambda(\delta_\nu^\pm(\lambda)) = \pi_-^\lambda(\delta_\nu^\pm(\lambda)) = 0,$$

for $\lambda \in \Omega_\nu^\pm$, and

$$U_\nu^\pm, L_\nu^\pm \text{ are rational in } \lambda \in \Omega_\nu^\pm, \text{ holomorphic in } \lambda \in (\Omega_\nu^\pm)^c,$$

$$U_\nu^\pm, L_\nu^\pm \text{ tends to } 0 \text{ as } |\lambda| \rightarrow \infty.$$

Besides,

$$\begin{aligned}
b(x)\tilde{\eta}_\nu^\mp(x, \lambda) &= \sigma_1(b(x)\eta_\nu^\pm(x, 1/\lambda)(1 + U_\nu^\pm(0))), \\
b(x)\tilde{\rho}_\nu^\mp(x, \lambda) &= \sigma_1(b(x)\rho_\nu^\pm(x, 1/\lambda)(1 + L_\nu^\pm(0))\delta_\nu^\pm(0)), \\
1 + \tilde{U}_\nu^\mp(\lambda) &= \sigma_1((1 + U_\nu^\pm(0))^{-1}(1 + U_\nu^\pm(1/\lambda))), \\
1 + \tilde{L}_\nu^\mp(\lambda) &= \sigma_1(\delta_\nu^\pm(0)^{-1}(1 + L_\nu^\pm(0))^{-1}(1 + L_\nu^\pm(1/\lambda))\delta_\nu^\pm(0)), \\
\tilde{\delta}_\nu^\mp(\lambda) &= \sigma_1(\delta_\nu^\pm(0)^{-1}\delta_\nu^\pm(1/\lambda)).
\end{aligned}$$

Finally, if we denote by m_+ the limits on Σ_a from components $\Omega_0^+, \Omega_1^-, \Omega_2^+, \Omega_3^-, \dots$ and from $D_0^+, D_1^-, D_2^+, D_3^-, \dots$, and denote by m_- the limits from the other components, then

$$m_+(x, \lambda) = m_-(x, \lambda)e^{-x(\lambda a + \frac{1}{\lambda}\sigma_1(a))}V(\lambda)e^{x(\lambda a + \frac{1}{\lambda}\sigma_1(a))} \quad \text{for } \lambda \in \Sigma_a. \quad (4.26)$$

Proof. A refined argument of the proof of Theorem 4.1 can derive the factorization of $m(x, \lambda)$ on $\mathbb{R} \times \Omega_\nu^\pm$. The jump condition (4.26) comes from the limits \tilde{m}_\pm , defined by (4.2), exist on Σ_a , \tilde{m}_\pm satisfy the same equation (4.4) and the operator $\partial_x - \mathcal{J}_\lambda$ is a derivation. \square

Definition 4.4. If the assumption in Corollary 4.1 holds, then (U_ν^\pm, V) is called the associated scattering data of the potential (b, v) .

Definition 4.5. Let $P(\lambda)$ be the matrix satisfying that $P^{-1}\mathcal{J}_\lambda P$ is a diagonal matrix with decreasing entries and $P_\pm = \lim_{\lambda_n \rightarrow \lambda} P(\lambda_n)$, $\lambda_n \in \Omega_0^\pm \cup \Omega_1^\mp \cup \Omega_2^\pm \cup \dots \cup D_0^\pm \cup D_1^\mp \cup \dots$.

Note that P is constant on each component of \mathbb{C}/Σ_a .

Theorem 4.2. Let $a \in \mathcal{A}$, a constant oblique direction. Suppose $b(x) \in K'_0$, $v(x) \in \mathcal{S}_0$, and their derivatives are rapidly decreasing as $|x| \rightarrow \infty$. If the set Z in Theorem 4.1 is a finite set contained in $\mathbb{C} \setminus \Sigma_a$. Then for the scattering data (U_ν^\pm, V) , we have the analytical constraints

$$\partial_\lambda^\alpha (V - I) \text{ is } O(\lambda^N) \text{ as } \lambda \rightarrow 0 \text{ and } O(\lambda^{-N}) \text{ as } \lambda \rightarrow \infty \text{ for } N, \alpha \geq 0, \quad (4.27)$$

$$\text{the product of limits of } V \text{ from each component, arranged clockwise,} \quad (4.28)$$

is I at each intersection of Σ_a ;

$$U_\nu^\pm \text{ are rational in } \lambda \in \Omega_\nu^\pm, \text{ holomorphic in } \lambda \in (\Omega_\nu^\pm)^c, \quad (4.29)$$

$$U_\nu^\pm \text{ tend to } 0 \text{ as } |\lambda| \rightarrow \infty. \quad (4.30)$$

the algebraic constraints

$$\pi_0^\lambda(U_\nu^\pm(\lambda)) = \pi_-^\lambda(U_\nu^\pm(\lambda)) = 0, \quad (4.31)$$

$$\tilde{J}(1 + U_\nu^\pm(\bar{\lambda}))^* \tilde{J}(1 + U_\nu^\pm(\lambda)) = 1, \quad (U_\nu^\pm(\bar{\lambda}))^* = (U_\nu^\pm)^t(\lambda), \quad (4.32)$$

$$\sigma_0(U_\nu^\pm(-\lambda)) = U_\nu^\mp(\lambda), \quad (4.33)$$

for $\lambda \in \Omega_\nu^\pm$ and

$$d_k^+(P_+^{-1}VP_+) = 1, \quad d_k^-(P_+^{-1}VP_+) \neq 0, \quad (4.34)$$

$$\tilde{J}V(\bar{\lambda})^* \tilde{J}V(\lambda) = 1, \quad (V(\bar{\lambda}))^* = V^t(\lambda), \quad (4.35)$$

$$\sigma_0(V(-\lambda))V(\lambda) = 1, \quad (4.36)$$

$$\sigma_1(V(1/\lambda))V(\lambda) = 1 \quad (4.37)$$

for $\lambda \in \Sigma_a$, $\forall 1 \leq k \leq 2n$. Here $d_k^\pm(f)$ denote the upper and lower $k \times k$ minors of f .

Proof. Properties (4.29)-(4.31) have been shown in Corollary 4.1. The analytic constraints (4.27), and (4.28) can be deduced from the results of [4]. Note (4.10) implies

$$\tilde{J}m(x, \bar{\lambda})^* \tilde{J}m(x, \lambda) = 1, \quad m(x, \bar{\lambda})^* = m^t(x, \lambda), \quad \sigma_0(m(x, -\lambda)) = m(x, \lambda). \quad (4.38)$$

Together with the uniqueness of the factorization of $m(x, \lambda)$ on $\mathbb{R} \times \Omega_\nu^\pm$, we derive the reality conditions (4.32), (4.33). Similarly, (4.26) implies

$$\begin{aligned} \tilde{J}m_+(x, \bar{\lambda})^* \tilde{J} &= \tilde{J}e^{x(\lambda a + \frac{1}{\lambda}\sigma_1(a))} V(\bar{\lambda})^* e^{-x(\lambda a + \frac{1}{\lambda}\sigma_1(a))} m_-(x, \bar{\lambda})^* \tilde{J} \\ m_+(x, \bar{\lambda})^* &= e^{x(\lambda a + \frac{1}{\lambda}\sigma_1(a))} V(\bar{\lambda})^* e^{-x(\lambda a + \frac{1}{\lambda}\sigma_1(a))} m_-(x, \bar{\lambda})^* \end{aligned}$$

for $\lambda \in \Sigma_a$. Hence

$$\begin{aligned} m_+(x, \lambda)^{-1} &= e^{-x(\lambda a + \frac{1}{\lambda}\sigma_1(a))} \tilde{J}V(\bar{\lambda})^* \tilde{J}e^{x(\lambda a + \frac{1}{\lambda}\sigma_1(a))} m_-(x, \lambda)^{-1} \\ m_+(x, \lambda)^t &= e^{x(\lambda a + \frac{1}{\lambda}\sigma_1(a))} V(\bar{\lambda})^* e^{-x(\lambda a + \frac{1}{\lambda}\sigma_1(a))} m_-(x, \lambda)^t \end{aligned}$$

for $\lambda \in \Sigma_a$ by (4.38). So (4.35) is proved. On the other hand, applying σ_0 to both sides of (4.26) and using (4.38), we obtain

$$m_-(x, -\lambda) = m_+(x, -\lambda))e^{x(\lambda a + \frac{1}{\lambda}\sigma_1(a))} \sigma_0(V(\lambda))e^{-x(\lambda a + \frac{1}{\lambda}\sigma_1(a))} \quad \text{for } \lambda \in \Sigma_a.$$

Hence we justify (4.36). Finally (4.37) is proved by applying σ_1 to both sides of (4.26) and using (4.11) instead.

To prove (4.34), we compute

$$\begin{aligned} & d_k^+(P_+^{-1}VP_+)(\lambda) \\ &= d_k^+(P_+^{-1}(m_-^{-1}m^+)(0, \lambda)P_+) \\ &= d_k^+(P_+^{-1}(1 + U_-)^{-1}\eta_-(0, \lambda)^{-1}\eta_+(0, \lambda)(1 + U_+)P_+) \\ &= 1 \end{aligned}$$

by (4.25), (4.31), $\lim_{x \rightarrow -\infty} \eta_\nu^\pm = \lim_{x \rightarrow -\infty} b = 1$, and $\tilde{\eta}_\nu^\pm(x, \lambda)$ satisfying (4.4) [23]. Here

$$U(\lambda) = \begin{cases} U_\nu^\pm(\lambda), & \lambda \in \Omega_\nu^\pm, \\ \tilde{U}_\nu^\pm(\lambda), & \lambda \in D_\nu^\pm, \end{cases} \quad \eta(x, \lambda) = \begin{cases} \eta_\nu^\pm(x, \lambda), & \lambda \in \Omega_\nu^\pm, \\ \tilde{\eta}_\nu^\pm(x, \lambda), & \lambda \in D_\nu^\pm, \end{cases}$$

and f_\pm the limits on Σ_a from $\Omega_0^\pm \cup \Omega_1^\mp \cup \Omega_2^\pm \cup \Omega_3^\mp \cup \dots \cup D_0^\pm \cup D_1^\mp \cup D_2^\pm \cup D_3^\mp \cup \dots$. $d_k^-(P_+^{-1}VP_+) \neq 0$ can be proved by analogy. \square

Remark 3. For $i \in \{0, 1, \dots, n-1\}$, replacing σ_1 by σ_i (defined in Remark 1), we can solve the associated direct problem by analogy.

5 The inverse scattering problem I

The goal of the inverse problem is to find the potential (b, v) for a given scattering data (U_ν^\pm, V) . Usually we try to reverse the process in the direct problem. However there exists

technical difficulties to find m since the boundary value of m is $b(x)$ as $\lambda \rightarrow \infty$. Hence it is impossible for us to pose the Riemann-Hilbert problem for m .

We will adopt the approach of [23] to construct a normalized eigenfunction $\check{m}(x, \lambda)$ prescribing the given scattering data (U_ν^\pm, V) with boundary value 1 at infinity in this section. Then we will try to find a gauge which transforms \check{m} into a solution $m(x, \lambda)$ of (4.3) in next section. We note in [1], [5], they solve the inverse problem for the twisted $\frac{O(n, n)}{O(n) \times O(n)}$ -system with U^\pm admitting only simple poles.

Theorem 5.1. *Let $q = 0$ and $a \in \mathcal{A}$. Suppose (U_ν^\pm, V) satisfies (4.27)-(4.37). Then there exists uniquely an $\check{m}(x, \lambda) \in L^{\sigma_0}$ satisfying*

$$\partial_x^{k'} (\check{m}(x, \lambda) - I) \in L_2^k(\Sigma_a) \text{ for } \forall k, k', \text{ and tends to 0 uniformly as } x \rightarrow -\infty, \quad (5.1)$$

$$\check{m}(x, \lambda) = \begin{cases} \eta_\nu^\pm(x, \lambda) e^{x\mathcal{J}_\lambda} (1 + U_\nu^\pm(\lambda)), & (x, \lambda) \in \{x \leq 0\} \times \Omega_\nu^\pm, & (1) \\ \rho_\nu^\pm(x, \lambda) e^{x\mathcal{J}_\lambda} (1 + L_\nu^\pm(\lambda)) \delta_\nu^\pm(\lambda), & (x, \lambda) \in \{x \geq 0\} \times \Omega_\nu^\pm, & (2) \\ \tilde{\eta}_\nu^\pm(x, \lambda) e^{x\mathcal{J}_\lambda} (1 + \tilde{U}_\nu^\pm(\lambda)), & (x, \lambda) \in \{x \leq 0\} \times D_\nu^\pm, & (3) \\ \tilde{\rho}_\nu^\pm(x, \lambda) e^{x\mathcal{J}_\lambda} (1 + \tilde{L}_\nu^\pm(\lambda)) \tilde{\delta}_\nu^\pm(\lambda), & (x, \lambda) \in \{x \geq 0\} \times D_\nu^\pm, & (4) \end{cases} \quad (5.2)$$

and

$$\check{m}_+(x, \lambda) = \check{m}_-(x, \lambda) e^{-x(\lambda a + \frac{1}{\lambda} \sigma_1(a))} V(\lambda) e^{x(\lambda a + \frac{1}{\lambda} \sigma_1(a))} \text{ for } \lambda \in \Sigma_a. \quad (5.3)$$

Here

$$\eta_\nu^\pm(x, \lambda), \text{ and } \rho_\nu^\pm(x, \lambda) \text{ are holomorphic and uniformly bounded,} \quad (5.4)$$

$$\eta_\nu^\pm(x, \lambda), \text{ and } \rho_\nu^\pm(x, \lambda) \text{ tend to 1 as } |\lambda| \rightarrow \infty, \quad (5.5)$$

$$\delta_\nu^\pm(\lambda), \tilde{\delta}_\nu^\pm(\lambda) \text{ are meromorphic, and tend to 1 as } |\lambda| \rightarrow \infty, \quad (5.6)$$

$$\pi_0^\lambda(L_\nu^\pm(\lambda)) = \pi_+^\lambda(L_\nu^\pm(\lambda)) = 0, \quad (5.7)$$

$$\pi_+^\lambda(\delta_\nu^\pm(\lambda)) = \pi_-^\lambda(\delta_\nu^\pm(\lambda)) = 0, \quad (5.8)$$

for $\lambda \in \Omega_\nu^\pm$,

$$\tilde{\eta}_\nu^\pm(x, \lambda), \text{ and } \tilde{\rho}_\nu^\pm(x, \lambda) \text{ are holomorphic and uniformly bounded} \quad (5.9)$$

for $\lambda \in D_\nu^\pm$, and

$$L_\nu^\pm \text{ are rational in } \lambda \in \Omega_\nu^\pm, \text{ holomorphic in } \lambda \in (\Omega_\nu^\pm)^c, \quad (5.10)$$

$$L_\nu^\pm \text{ tends to 0 as } |\lambda| \rightarrow \infty, \quad (5.11)$$

$$1 + \tilde{U}_\nu^\mp(\lambda) = \sigma_1((1 + U_\nu^\pm(0))^{-1}(1 + U_\nu^\pm(1/\lambda))), \quad (5.12)$$

$$1 + \tilde{L}_\nu^\mp(\lambda) = \sigma_1(\delta_\nu^\pm(0)^{-1}(1 + L_\nu^\pm(0))^{-1}(1 + L_\nu^\pm(1/\lambda))\delta_\nu^\pm(0)), \quad (5.13)$$

$$\tilde{\delta}_\nu^\mp(\lambda) = \sigma_1(\delta_\nu^\pm(0)^{-1}\delta_\nu^\pm(1/\lambda)). \quad (5.14)$$

Proof. Step 1: $\{x \leq 0\} \times (\mathbb{C} \setminus \Sigma_a)$

Define \tilde{U}_ν^\pm by (5.12). Let

$$U(\lambda) = \begin{cases} U_\nu^\pm(\lambda), & \lambda \in \Omega_\nu^\pm, \\ \tilde{U}_\nu^\pm(\lambda), & \lambda \in D_\nu^\pm, \end{cases} \quad (5.15)$$

and U_{\pm} the limits on Σ_a from $\Omega_0^{\pm} \cup \Omega_1^{\mp} \cup \Omega_2^{\pm} \cup \Omega_3^{\mp} \cup \dots \cup D_0^{\pm} \cup D_1^{\mp} \cup D_2^{\pm} \cup D_3^{\mp} \cup \dots$. Define

$$W(\lambda) = (1 + U_{-}(\lambda))V(\lambda)(1 + U_{+}(\lambda))^{-1} \quad \text{for } \lambda \in \Sigma_a. \quad (5.16)$$

Then to show (1), (3) in (5.2), and the statement about η_{ν}^{\pm} in (5.4), (5.5) is equivalent to solving the following Riemann-Hilbert problem with purely continuous scattering data

$$f_{+}(x, \lambda) = f_{-}(x, \lambda)e^{-x(\lambda a + \frac{1}{\lambda}\sigma_1(a))}W(\lambda)e^{x(\lambda a + \frac{1}{\lambda}\sigma_1(a))} \quad \text{for } \lambda \in \Sigma_a, \quad (5.17)$$

$$f \text{ is holomorphic in } \mathbb{C}/\Sigma_a, \quad f(x, \lambda) \rightarrow 1, \text{ as } \lambda \rightarrow \infty \quad (5.18)$$

and setting

$$f(x, \lambda) = \begin{cases} \eta_{\nu}^{\pm}(x, \lambda), & \lambda \in \Omega_{\nu}^{\pm}, \\ \tilde{\eta}_{\nu}^{\pm}(x, \lambda), & \lambda \in D_{\nu}^{\pm}. \end{cases}$$

To solve the above Riemann-Hilbert problem, one can apply the method (§10 in [4]) to prove its Fredholm property. So the solvability is reduced to showing that the homogeneous solution is trivial. Let $g(x, \lambda) = f(x, \lambda)f(x, -\bar{\lambda})^*$ and f satisfies (5.17), is holomorphic in \mathbb{C}/Σ_a , and tends to 0 as $\lambda \rightarrow \infty$. Hence if

$$W(-\bar{\lambda})^* = W(\lambda), \quad \lambda \in \Sigma_a, \quad (5.19)$$

then for $\lambda \in \Sigma_a$,

$$\begin{aligned} g_{-}(x, \lambda) &= f_{-}(x, \lambda) (f_{+}(x, -\bar{\lambda}))^* \\ &= f_{-}(x, \lambda) \left(f_{-}(x, -\bar{\lambda}) e^{x(\bar{\lambda} a + \frac{1}{\bar{\lambda}}\sigma_1(a))} W(-\bar{\lambda}) e^{-x(\bar{\lambda} a + \frac{1}{\bar{\lambda}}\sigma_1(a))} \right)^* \\ &= f_{-}(x, \lambda) (e^{x\mathcal{J}} W(\lambda)) f_{-}(x, -\bar{\lambda})^* \\ &= f_{+}(x, \lambda) f_{-}(x, -\bar{\lambda})^* \\ &= g_{+}(x, \lambda). \end{aligned}$$

So $g = 0$ by the Liouville's theorem. Moreover, the positivity condition (5.19) implies $f = 0$ for $\lambda \in i\mathbb{R}$. Hence $f \equiv 0$ by holomorphicity.

To prove the claim (5.19), we note that the algebraic constraints (4.32), (4.33), (4.35) and (4.36) of the scattering data imply

$$V(-\bar{\lambda})^* = V(\lambda), \quad (1 + U_{\pm}(-\bar{\lambda}))^* = (1 + U_{\mp}(\lambda))^{-1}. \quad (5.20)$$

Using these reality conditions and the definition of W , we can derive (5.19).

Step 2: Construction of δ_{ν}^{\pm} , $\tilde{\delta}_{\nu}^{\pm}$, L_{ν}^{\pm} , \tilde{L}_{ν}^{\pm}

We first solve the scalar Riemann-Hilbert problems for the entries of a matrix $\Delta(\lambda)$:

$$\Delta \text{ is meromorphic in } \mathbb{C}/\Sigma_a, \quad \Delta(\lambda) \rightarrow 1 \text{ as } \lambda \rightarrow \infty, \quad (5.21)$$

$$\pi_{+}^{\lambda}(\Delta(\lambda)) = \pi_{-}^{\lambda}(\Delta(\lambda)) = 0, \quad (5.22)$$

$$[P_{+}^{-1}\Delta_{+}P_{+}]_k = [P_{+}^{-1}\Delta_{-}P_{+}]_k \frac{d_{2n-k+1}^{-}(P_{+}^{-1}VP_{+})}{d_{2n-k}^{-}(P_{+}^{-1}VP_{+})}, \quad (5.23)$$

$$\{ \text{zeros of } [P^{-1}\Delta P]_k \} = \{ \text{poles of the } k+1\text{-th column of } P^{-1}UP \}, \quad (5.24)$$

$$\{ \text{poles of } [P^{-1}\Delta P]_k \} = \{ \text{poles of the } k\text{-th column of } P^{-1}UP \}. \quad (5.25)$$

Here P is defined by Definition 4.5, $[f]_k$ denotes the k -th entry of the diagonal part of f , for $1 \leq k \leq 2n$. Then δ_ν^\pm , and $\tilde{\delta}_\nu^\pm$ are constructed by

$$\Delta(\lambda) = \begin{cases} \delta_\nu^\pm(\lambda), & \lambda \in \Omega_\nu^\pm, \\ \tilde{\delta}_\nu^\pm(\lambda), & \lambda \in D_\nu^\pm. \end{cases} \quad (5.26)$$

To construct L_ν^\pm , \tilde{L}_ν^\pm , by (4.34), (5.22) and (5.23), one can factorize

$$V(\lambda) = (1 + \hat{L}^-(\lambda))^{-1} \Delta_-^{-1}(\lambda) \Delta_+(\lambda) (1 + \hat{L}^+(\lambda)), \quad (5.27)$$

$$\pi_+^{\lambda_n^+}(\hat{L}^+) = \pi_0^{\lambda_n^+}(\hat{L}^+) = 0,$$

$$\pi_+^{\lambda_n^-}(\hat{L}^-) = \pi_0^{\lambda_n^-}(\hat{L}^-) = 0,$$

on Σ_a . Here $\lambda_n^\pm \rightarrow \lambda$, $\lambda_n^\pm \in \Omega_0^\pm \cup \Omega_1^\mp \cup \Omega_2^\pm \cup \Omega_3^\mp \cup \dots \cup D_0^\pm \cup D_1^\mp \cup D_2^\pm \cup D_3^\mp \cup \dots$. By Theorem 5.2 of [23], we can have the extensions $L^{\pm,0}$, $R^{\pm,0}$ satisfying

$$\Delta_\pm(\lambda)(1 + \hat{L}^\pm(\lambda))\Delta_\pm^{-1}(\lambda) = (1 + R^\pm(\lambda))(1 + L^\pm(\lambda)) \text{ on } \Sigma_a, \quad (5.28)$$

$$\pi_+^{\lambda_n^\pm}(R^\pm) = \pi_0^{\lambda_n^\pm}(R^\pm) = 0, \quad \pi_+^{\lambda_n^\pm}(L^\pm) = \pi_0^{\lambda_n^\pm}(L^\pm) = 0 \text{ on } \mathcal{O}^\pm, \quad (5.29)$$

$$R^\pm \text{ is holomorphic on } \mathcal{O}^\pm, \text{ meromorphic on } (\mathcal{O}^\pm)^c, \quad (5.30)$$

$$L^\pm \text{ is holomorphic on } (\mathcal{O}^\pm)^c, \text{ rational on } \mathcal{O}^\pm. \quad (5.31)$$

Here \mathcal{O}^\pm is the component of $\Omega_0^\pm \cup \Omega_1^\mp \cup \Omega_2^\pm \cup \Omega_3^\mp \cup \dots \cup D_0^\pm \cup D_1^\mp \cup D_2^\pm \cup D_3^\mp \cup \dots$ which contains λ_n^\pm , $\lambda_n^\pm \rightarrow \lambda$. Hence (5.27), (5.28) imply

$$V(\lambda) = \Delta_-^{-1}(\lambda)(1 + L_-(\lambda))^{-1} R(\lambda)(1 + L_+(\lambda)) \Delta_+(\lambda). \quad (5.32)$$

Where Δ is defined by (5.26),

$$R = (1 + R^-(\lambda))^{-1} (1 + R^+(\lambda))$$

$$L(\lambda) = \begin{cases} L^+(\lambda), & \lambda \in \mathcal{O}^+, \\ L^-(\lambda), & \lambda \in \mathcal{O}^-. \end{cases} \quad (5.33)$$

Finally, $L_\nu^\pm(\lambda)$, $\tilde{L}_\nu^\pm(\lambda)$ are defined by

$$L(\lambda) = \begin{cases} L_\nu^\pm(\lambda), & \lambda \in \Omega_\nu^\pm, \\ \tilde{L}_\nu^\pm(\lambda), & \lambda \in D_\nu^\pm. \end{cases} \quad (5.34)$$

Step 3: $\{x \geq 0\} \times (\mathbb{C} \setminus \Sigma_a)$

Having constructed δ_ν^\pm , $\tilde{\delta}_\nu^\pm$, L_ν^\pm , \tilde{L}_ν^\pm , one can adopt the scheme of Step 1 to find ρ_ν^\pm and $\tilde{\rho}_\nu^\pm$. More precisely, let

$$\Lambda(\lambda) = (1 + L_-(\lambda)) \Delta_-(\lambda) V(\lambda) \Delta_+^{-1}(\lambda) (1 + L_+(\lambda))^{-1} \text{ for } \lambda \in \Sigma_a.$$

Then to show (2), (4) in (5.2), and the statement about ρ_ν^\pm in (5.4), (5.5) is equivalent to solving the following Riemann-Hilbert problem with purely continuous scattering data

$$f_+(x, \lambda) = f_-(x, \lambda) e^{-x(\lambda a + \frac{1}{\lambda} \sigma_1(a))} \Lambda(\lambda) e^{x(\lambda a + \frac{1}{\lambda} \sigma_1(a))} \text{ for } \lambda \in \Sigma_a, \quad (5.35)$$

$$f \text{ is holomorphic in } \mathbb{C} \setminus \Sigma_a, \quad f(x, \lambda) \rightarrow 1, \text{ as } \lambda \rightarrow \infty \quad (5.36)$$

and setting

$$f(x, \lambda) = \begin{cases} \rho_\nu^\pm(x, \lambda), & \lambda \in \Omega_\nu^\pm, \\ \tilde{\rho}_\nu^\pm(x, \lambda), & \lambda \in D_\nu^\pm. \end{cases}$$

As above, the solvability can be reduced to showing:

$$\Lambda(-\bar{\lambda})^* = \Lambda(\lambda), \quad \lambda \in \Sigma_a, \quad (5.37)$$

and it can be implied by the reality conditions

$$V(-\bar{\lambda})^* = V(\lambda), \quad \Delta_\pm(-\bar{\lambda})^* = \Delta_\mp(\lambda)^{-1}, \quad (1 + L_\pm(-\bar{\lambda}))^* = (1 + L_\mp(\lambda))^{-1}. \quad (5.38)$$

The reality condition $V(-\bar{\lambda})^* = V(\lambda)$ follows from the algebraic constraints (4.35), (4.36). Moreover, by $V(-\bar{\lambda})^* = V(\lambda)$, (5.27), we have

$$V(\lambda) = (1 + \hat{L}^+(-\bar{\lambda})^*) \Delta_+^*(-\bar{\lambda}) (\Delta_-^*(-\bar{\lambda}))^{-1} ((1 + \hat{L}^-(-\bar{\lambda})^*)^{-1})$$

Together with (4.34), we conclude $(\Delta(-\bar{\lambda})^*)^{-1}$ satisfies (5.23). So there is no jump across Σ_a for $\Delta(-\bar{\lambda})^* \Delta(\lambda)$. On the other hand, by (5.24), (4.32), and (4.33),

$$\begin{aligned} & \left\{ \text{zeros of } \left[P^{-1}(\lambda) (\Delta(-\bar{\lambda})^*)^{-1} P(\lambda) \right]_k \right\} \\ = & \left\{ \text{zeros of } \left[P^{-1}(\lambda) (P(-\bar{\lambda})^*)^{-1} P(-\bar{\lambda})^* (\Delta(-\bar{\lambda})^*)^{-1} (P(-\bar{\lambda})^*)^{-1} P(-\bar{\lambda})^* P(\lambda) \right]_k \right\} \\ = & \left\{ \text{zeros of } \left[P(-\bar{\lambda})^* (\Delta(-\bar{\lambda})^*)^{-1} (P(-\bar{\lambda})^*)^{-1} \right]_{2n-k+1} \right\} \\ = & \left\{ \text{poles of the } 2n - k\text{-th column of } P^*(-\bar{\lambda}) U(\lambda) P(-\bar{\lambda}) \right\} \\ = & \left\{ \text{poles of the } k + 1\text{-th column of } P(\lambda)^{-1}(\lambda) U(\lambda) P(\lambda) \right\} \\ = & \left\{ \text{zeros of } [P^{-1}(\lambda) \Delta(\lambda) P(\lambda)]_k \right\} \end{aligned}$$

Similarly, (5.25), (4.32), and (4.33) imply

$$\left\{ \text{poles of } [P^{-1}(\lambda) (\Delta(-\bar{\lambda})^*)^{-1} P(\lambda)]_k \right\} = \left\{ \text{poles of } [P^{-1}(\lambda) \Delta(\lambda) P(\lambda)]_k \right\}$$

Thus the Liouville's theorem implies $\Delta(-\bar{\lambda})^* = \Delta(\lambda)^{-1}$ and $(1 + \hat{L}^\pm(-\bar{\lambda}))^* = (1 + \hat{L}^\mp(\lambda))^{-1}$. Finally

$$(1 + L^{\pm,0}(-\bar{\lambda}))^* = (1 + L^{\mp,0}(\lambda))^{-1} \quad (5.39)$$

by $V(-\bar{\lambda})^* = V(\lambda)$, (5.27), (5.28), (5.30), (5.31), and the Liouville's theorem. So $(1 + L_\pm(-\bar{\lambda}))^* = (1 + L_\mp(\lambda))^{-1}$ comes from (5.33), (5.39), and $\Delta_\pm(-\bar{\lambda})^* = \Delta_\mp(\lambda)^{-1}$.

Step 4: Proof of $\check{m}(x, \lambda) \in L^{\sigma_0}$

Let $W(\lambda)$ be defined by (5.16), then

$$\tilde{J}W(\bar{\lambda})^* \tilde{J}W(\lambda) = 1, \quad (W(\bar{\lambda}))^* = W^t(\lambda), \quad (5.40)$$

$$\sigma_0(W(-\lambda)) W(\lambda) = 1, \quad (5.41)$$

by (4.32), (4.33), (4.35), and (4.36). Therefore, one can reverse the process in the proof of Theorem 4.2 to show

$$\tilde{J}\check{m}(x, \bar{\lambda})^* \tilde{J}\check{m}(x, \lambda) = 1, \quad \check{m}(x, \bar{\lambda})^* = \check{m}^t(x, \lambda), \quad \sigma_0(\check{m}(x, -\lambda)) = \check{m}(x, \lambda). \quad (5.42)$$

So $\check{m}(x, \lambda) \in L^{\sigma_0}$ for $x \leq 0$.

As for $x \geq 0$, note the formula (1)-(4), (5.42), and the unique factorization properties imply

$$\begin{aligned}\tilde{J}(1 + L_\nu^+(\bar{\lambda}))^* \tilde{J}(1 + L_\nu^+(\lambda)) &= 1, \quad (L_\nu^\pm(\bar{\lambda}))^* = (L_\nu^\pm)^t(\lambda), \\ \sigma_0(L_\nu^-(-\lambda)) &= L_\nu^+(\lambda), \\ \tilde{J}\delta_\nu^+(\bar{\lambda})^* \tilde{J}\delta_\nu^+(\lambda) &= 1, \quad \delta_\nu^\pm(\bar{\lambda})^* = (\delta_\nu^\pm)^t(\lambda), \\ \sigma_0(\delta_\nu^-(-\lambda)) &= \delta_\nu^+(\lambda).\end{aligned}$$

Therefore, using the above argument, one can justify $\check{m}(x, \lambda) \in L^{\sigma_0}$ for $x \geq 0$.

Step 5: Proof of (5.13) and (5.14)

The strategy of the proof is analogous to that of showing (5.38). By (4.35), (4.37), (5.27), we have

$$V(\lambda) = (1 + \sigma_1(\hat{L}^+(\frac{1}{\lambda})))^{-1} \sigma_1(\Delta_+(\frac{1}{\lambda}))^{-1} \sigma_1((\Delta_-(\frac{1}{\lambda}))(1 + \sigma_1(\hat{L}^-(\frac{1}{\lambda}))))^{-1} \quad (5.43)$$

Thus $\sigma(\Delta(\frac{1}{\lambda}))$ satisfies (5.23) by (4.34) and $\sigma(\Delta(\frac{1}{\lambda}))^{-1} \Delta(\lambda)$ has no jump across Σ_a . On the other hand, by (5.24), (5.12), for $\lambda \in \mathbb{C} \setminus \Sigma_a$,

$$\begin{aligned}& \left\{ \text{zeros of } \left[P(\lambda)^{-1} \sigma_1(\Delta(\frac{1}{\lambda})) P(\lambda) \right]_k \right\} \\ &= \left\{ \text{zeros of } \left[P(\lambda)^{-1} \sigma_1(P(\frac{1}{\lambda}) P(\frac{1}{\lambda})^{-1} \Delta(\frac{1}{\lambda}) P(\frac{1}{\lambda}) P(\frac{1}{\lambda})^{-1} P(\lambda)) \right]_k \right\} \\ &= \left\{ \text{zeros of } \left[P(\frac{1}{\lambda})^{-1} \Delta(\frac{1}{\lambda}) P(\frac{1}{\lambda}) \right]_k \right\} \\ &= \left\{ \text{poles of the } k+1\text{-th column of } P(\frac{1}{\lambda})^{-1} U(\frac{1}{\lambda}) P(\frac{1}{\lambda}) \right\} \\ &= \left\{ \text{poles of the } k+1\text{-th column of } P(\frac{1}{\lambda})^{-1} \sigma_1(U(\lambda)) P(\frac{1}{\lambda}) \right\} \\ &= \left\{ \text{poles of the } k+1\text{-th column of } P(\lambda)^{-1} U(\lambda) P(\lambda) \right\} \\ &= \left\{ \text{zeros of } [P^{-1}(\lambda) \Delta(\lambda) P(\lambda)]_k \right\}\end{aligned}$$

Here we have use the diagonal property of $P(\lambda)^{-1} \sigma_1(P(\frac{1}{\lambda}))$. Similarly,

$$\left\{ \text{poles of } [P^{-1}(\lambda) \sigma_1(\Delta(\frac{1}{\lambda})) P(\lambda)]_k \right\} = \left\{ \text{poles of } [P^{-1}(\lambda) \Delta(\lambda) P(\lambda)]_k \right\}$$

can be justified by (5.25), (5.12). Thus the Liouville's theorem implies $\sigma_1(\Delta(\frac{1}{\lambda}))^{-1} \Delta(\lambda)$ is a constant. Hence we prove (5.14) by a normalization. Furthermore, $\sigma_1(\hat{L}^\pm(\frac{1}{\lambda}))$ are equal to $\hat{L}^\mp(\lambda)$ up to a constant by (5.43). So (5.13) follows by (5.28), (5.30), (5.31), (5.33), and (5.14).

The analytic constraints (5.1) can be deduced from the results of [4].

□

Theorem 5.2. *Let $q \neq 0$ and $a \in \mathcal{A}$. Suppose V satisfies (4.27), (4.28), (4.34)-(4.37) and $|V - 1|_{L^\infty} \ll 1$. Then there exists uniquely an $\check{m}(x, \lambda) \in L^{\sigma_0}$ satisfying*

$$\check{m}_+(x, \lambda) = \check{m}_-(x, \lambda) e^{-x(\lambda a + \frac{1}{\lambda} \sigma_1(a))} V(\lambda) e^{x(\lambda a + \frac{1}{\lambda} \sigma_1(a))} \quad \text{for } \lambda \in \Sigma_a, \quad (5.44)$$

$$\check{m} \text{ is holomorphic in } \mathbb{C}/\Sigma_a, \check{m}(x, \lambda) \rightarrow 1, \text{ as } \lambda \rightarrow \infty, \quad (5.45)$$

$$\partial_x^{k'} (\check{m}(x, \lambda) - I) \in L_2^k(\Sigma_a) \text{ for } \forall k, k', \text{ and tends to 0 uniformly} \quad (5.46)$$

as $x \rightarrow -\infty$.

Proof. If $q \neq 0$, then the positivity conditions (5.19), (5.37) fail in Step 1 and Step 3. However, the Riemann Hilbert problems (5.17)-(5.18), (5.21)-(5.23), and (5.35)-(5.36), with $U(\lambda) \equiv 0$, $L(\lambda) \equiv 0$, in Step 1-3 can be solved under the small data constraint $|V - 1|_{L^2} \ll 1$. \square

Remark 4. *We remark that if $\delta_\nu^\pm, \tilde{\delta}_\nu^\pm, L_\nu^\pm, \tilde{L}_\nu^\pm$ are constructed by Corollary 4.1 and define Δ by (5.26), (5.34), then (5.21)-(5.25), (5.32) are valid by adapting the argument in the proof of Theorem 4.6 and 4.7 in [23].*

6 The inverse scattering problem II

We adapt the argument in [1] to complete the process of reconstructing the operator (4.1) in this section. That is, we need to find a proper gauge $b(x) \in K'_0$ which transforms $\check{m}(x, \lambda)$ to the eigenfunction $m(x, \lambda)$ of (4.3).

Write $a = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}$, and $D = \text{diag}(w_1, \dots, w_n)$. Define

$$\vec{x} = (x_1, \dots, x_n) = x(w_1, \dots, w_n), \quad (6.1)$$

$$X = \sum_{i=1}^n x_i a_i, \quad a_i \text{ are defined by (3.2)}, \quad (6.2)$$

$$M(\vec{x}, \lambda) = \check{m}(x, a, \lambda) = \check{m}(x, \lambda). \quad (6.3)$$

Lemma 6.1. *Suppose $\check{m}(x, \lambda)$ is derived by Theorem 5.1 or 5.2. Then*

$$\frac{\partial M}{\partial x_j} = \left[M, \lambda a_j + \frac{1}{\lambda} \sigma_1(a_j) \right] + \frac{1}{\lambda} (\sigma_1(a_j) - B_j(\vec{x})) M - C_j(\vec{x}) M, \quad (6.4)$$

with

$$B_j(\vec{x}) \in \mathcal{P}_0 \cap C^\infty, \quad C_j(\vec{x}) \in \mathcal{K}_0 \cap C^\infty. \quad (6.5)$$

Proof. We are going to show that $\{(\partial_{x_j} + ad(\lambda a_j + \frac{1}{\lambda} \sigma_1(a_j))) M\} M^{-1}$ is holomorphic on $\mathbb{C} \setminus \{0\}$ and bounded at ∞ . Hence (6.4) follows immediately from the asymptotic expansions

$$M(\vec{x}, \lambda) \rightarrow 1 + \sum_{k=1}^{\infty} M_k(\vec{x}) \lambda^{-k} \quad \text{as } |\lambda| \rightarrow \infty, \quad (6.6)$$

$$M(\vec{x}, \lambda) \rightarrow \sum_{k=0}^{\infty} \tilde{M}_k(\vec{x}) \lambda^k \quad \text{as } |\lambda| \rightarrow 0, \quad (6.7)$$

and the conditions (6.5) come from $((\partial_{x_j} + ad(\lambda a_j + \frac{1}{\lambda} \sigma_1(a_j))) \check{m}) \check{m}^{-1} \in \mathcal{L}^{\sigma_0}$ by using $\check{m} \in L^{\sigma_0}$, $\lambda a_j + \frac{1}{\lambda} \sigma_1(a_j) \in \mathcal{L}_+^{\sigma_0}$, (5.1), and (5.46).

If $x \leq 0$ and $\lambda \in \Omega_\nu^\pm$, by (1) in (5.2), (6.1), and (6.3), then

$$\begin{aligned}
& \left\{ \left(\partial_{x_j} + ad(\lambda a_j + \frac{1}{\lambda} \sigma_1(a_j)) \right) M \right\} M^{-1} \\
= & \left\{ \left(\partial_{x_j} + ad(\lambda a_j + \frac{1}{\lambda} \sigma_1(a_j)) \right) \eta_\nu^\pm(\vec{x}, \lambda) e^{-\lambda X - \frac{1}{\lambda} \sigma_1(X)} (1 + U_\nu^\pm(\lambda)) e^{\lambda X + \frac{1}{\lambda} \sigma_1(X)} \right\} \\
& \cdot \left(\eta_\nu^\pm(\vec{x}, \lambda) e^{-\lambda X - \frac{1}{\lambda} \sigma_1(X)} (1 + U_\nu^\pm(\lambda)) e^{\lambda X + \frac{1}{\lambda} \sigma_1(X)} \right)^{-1} \\
= & \left\{ \left(\partial_{x_j} + ad(\lambda a_j + \frac{1}{\lambda} \sigma_1(a_j)) \right) \eta_\nu^\pm \right\} (\eta_\nu^\pm)^{-1}.
\end{aligned}$$

Here we have used $(\partial_{x_j} + ad(\lambda a_j + \frac{1}{\lambda} \sigma_1(a_j))) e^{-(\lambda X + \frac{1}{\lambda} \sigma_1(X))} (1 + U) e^{\lambda X + \frac{1}{\lambda} \sigma_1(X)} = 0$. Using (2), (3), (4) in (5.2) and the same argument, we will derive similar formula on other components. So $\{(\partial_{x_j} + ad(\lambda a_j + \frac{1}{\lambda} \sigma_1(a_j))) M\} M^{-1}$ is regular on $\mathbb{C} \setminus \Sigma_a$ by the properties of η_ν^\pm , $\tilde{\eta}_\nu^\pm$, ρ_ν^\pm , and $\tilde{\rho}_\nu^\pm$ in Theorem 5.1. Besides, by (5.3), (5.44), we derive

$$\begin{aligned}
& \left\{ \left(\partial_{x_j} + ad(\lambda a_j + \frac{1}{\lambda} \sigma_1(a_j)) \right) M_+ \right\} M_+^{-1} \\
= & \left\{ \left(\partial_{x_j} + ad(\lambda a_j + \frac{1}{\lambda} \sigma_1(a_j)) \right) (M_+ - M_-) \right\} M_+^{-1} \\
& + \left\{ \left(\partial_{x_j} + ad(\lambda a_j + \frac{1}{\lambda} \sigma_1(a_j)) \right) M_- \right\} M_+^{-1} \\
= & \left\{ \left(\partial_{x_j} + ad(\lambda a_j + \frac{1}{\lambda} \sigma_1(a_j)) \right) M_- \right\} (e^{-(\lambda X + \frac{1}{\lambda} \sigma_1(X))} V(\lambda) e^{\lambda X + \frac{1}{\lambda} \sigma_1(X)} - 1) M_+^{-1} \\
& + \left\{ \left(\partial_{x_j} + ad(\lambda a_j + \frac{1}{\lambda} \sigma_1(a_j)) \right) M_- \right\} M_+^{-1} \\
= & \left\{ \left(\partial_{x_j} + ad(\lambda a_j + \frac{1}{\lambda} \sigma_1(a_j)) \right) M_- \right\} (e^{-(\lambda X + \frac{1}{\lambda} \sigma_1(X))} V(\lambda) e^{\lambda X + \frac{1}{\lambda} \sigma_1(X)} - 1) M_+^{-1} \\
& + \left\{ \left(\partial_{x_j} + ad(\lambda a_j + \frac{1}{\lambda} \sigma_1(a_j)) \right) M_- \right\} M_+^{-1} \\
= & \left\{ \left(\partial_{x_j} + ad(\lambda a_j + \frac{1}{\lambda} \sigma_1(a_j)) \right) M_- \right\} M_-^{-1}
\end{aligned}$$

Therefore, $\{(\partial_{x_j} + ad(\lambda a_j + \frac{1}{\lambda} \sigma_1(a_j))) M\} M^{-1}$ is continuous at Σ_a . The uniform boundedness of $\{(\partial_{x_j} + ad(\lambda a_j + \frac{1}{\lambda} \sigma_1(a_j))) M\} M^{-1}$ at ∞ can be seen by (6.6). \square

Lemma 6.2. *The compatibility conditions of (6.4) are*

$$\partial_{x_j} C_i - \partial_{x_i} C_j - [C_i, C_j] = [B_i, a_j] - [B_j, a_i]. \quad (6.8)$$

Proof. Taking the derivative ∂_{x_i} of (6.4), we have

$$\begin{aligned}\partial_{x_i}\partial_{x_j}M &= \left[[M, \lambda a_i + \frac{1}{\lambda}\sigma_1(a_i)], \lambda a_j + \frac{1}{\lambda}\sigma_1(a_j) \right] \\ &+ \left[\frac{1}{\lambda}(\sigma_1(a_i) - B_i)M - C_iM, \lambda a_j + \frac{1}{\lambda}\sigma_1(a_j) \right] - \left(\frac{1}{\lambda}\frac{\partial B_j}{\partial x_i} + \frac{\partial C_j}{\partial x_i} \right)M \\ &+ \left(\frac{1}{\lambda}(\sigma_1(a_j) - B_j) - C_j \right) \left[M, \lambda a_i + \frac{1}{\lambda}\sigma_1(a_i) \right] \\ &+ \left(\frac{1}{\lambda}(\sigma_1(a_j) - B_j) - C_j \right) \left(\frac{1}{\lambda}(\sigma_1(a_i) - B_i)M - C_iM \right).\end{aligned}$$

Letting $|\lambda| \rightarrow \infty$ and applying (6.6), we obtain

$$C_j(\vec{x}) = [M_1(\vec{x}), a_j]. \quad (6.9)$$

Hence

$$\begin{aligned}&\partial_{x_i}\partial_{x_j}M|_{\lambda=\infty} \\ &= [\sigma_1(a_i) - B_i, a_j] - [C_iM_1, a_j] - \frac{\partial C_j}{\partial x_i} - C_j[M_1, a_i] + C_jC_i + \text{symm. terms} \\ &= -[B_i, a_j] - \frac{\partial C_j}{\partial x_i} + C_jC_i + a_jC_iM_1 + C_ja_iM_1 + \text{symm. terms} \\ &= -[B_i, a_j] - \frac{\partial C_j}{\partial x_i} + C_jC_i + [a_j, C_i]M_1 - [a_i, C_j]M_1 + \text{symm. terms} \\ &= -[B_i, a_j] - \frac{\partial C_j}{\partial x_i} + C_jC_i + \text{symm. terms}.\end{aligned}$$

Here “symm. terms” denote terms which are symmetric with respect to i, j and they may differ from each other. Also, in the above computation, we have also used

$$[a_i, C_j], \left[[M, \lambda a_i + \frac{1}{\lambda}\sigma_1(a_i)], \lambda a_j + \frac{1}{\lambda}\sigma_1(a_j) \right] \text{ are symmetric with respect to } i, j$$

which follow from (6.9). Therefore, the compatibility conditions of (6.4) are (6.8). \square

Lemma 6.3. *Suppose either of the assumption in Theorem 5.1 or 5.2 holds. Then there exists*

$$b(\vec{x}) \in K'_0 \cap C^\infty, \quad (6.10)$$

such that

$$b(x(w_1, \dots, w_n)) \rightarrow 1 \quad \text{as } x \rightarrow -\infty, \quad (6.11)$$

$$-bC_jb^{-1} + (\partial_jb)b^{-1} \in \mathcal{S}_0 \quad \text{for } \forall j. \quad (6.12)$$

Proof. By (6.8), and (6.5), to prove (6.10) and (6.12), we need only to show

$$[B_j, a_i] \in \mathcal{S}_0, \quad i \neq j. \quad (6.13)$$

First of all, let us claim

$$M(\vec{x}, \lambda) = \sigma_1(\tilde{M}_0^{-1} M(\vec{x}, \frac{1}{\lambda})). \quad (6.14)$$

Here \tilde{M}_0 is defined by (6.7). The assertion can be proved by the Liouville's theorem and conditions (4.37), (5.3), (5.12)-(5.14) and (5.2). Therefore, taking the limits of (6.14) at $\lambda = 0$, we derive

$$\tilde{M}_0 \cdot \sigma_1(\tilde{M}_0) = 1. \quad (6.15)$$

Note $\tilde{M}_0 \in K_0$ by $\check{m}(x, \lambda) \in L^{\sigma_0}$. Hence

$$\tilde{M}_0 = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}, \quad \text{with } f_i \in O(J) \quad (6.16)$$

and $O(J) = \{x \in GL_n(\mathbb{C}) | Jx^*Jx = Jx^tJx = 1\}$. Combining (6.15) and (6.16), we have $f_1^2 = 1$. Therefore the minimal polynomial of f_1 must be a divisor of $\lambda^2 - 1$. If $J = I$, then (6.16) implies f_1 is always diagonalizable. If $J \neq I$, and $|V - 1|_{L^2} < 1$, by continuity, f_1 is diagonalizable, too. Hence the minimal polynomial of f_1 must be $\lambda - 1$. Therefore, we conclude

$$\tilde{M}_0 \in K'_0. \quad (6.17)$$

Now let us equate the λ^{-1} -terms of (6.4) at $\lambda = 0$, we obtain

$$\tilde{M}_0 \sigma_1(a_j) - B_j \tilde{M}_0 = 0, \quad (6.18)$$

Plugging (6.17) into (6.18) and solving for B_j , one can justify (6.13). \square

Theorem 6.1. *Suppose either of the assumption in Theorem 5.1 or 5.2 holds. Let*

$$\Psi(x, \lambda) = b(\vec{x}) M(\vec{x}, \lambda) e^{-(\lambda X + \frac{1}{\lambda} \sigma_1(X))} \quad (6.19)$$

Here x, \vec{x}, X, M satisfy (6.1)-(6.3). Then

$$\frac{\partial \Psi}{\partial x} = -\lambda b a b^{-1} \Psi - \frac{1}{\lambda} \sigma_1(b a b^{-1}) \Psi - v \Psi,$$

with

$$\begin{aligned} b(x) &= b(x(w_1, \dots, w_n)) \in K'_0 \cap C^\infty, \\ v(x) &= \sum w_j (b C_j b^{-1} - (\partial_j b) b^{-1}) (x(w_1, \dots, w_n)) \in \mathcal{S}_0 \cap C^\infty, \end{aligned} \quad (6.20)$$

where $C_j, b(x(w_1, \dots, w_n))$ are defined by Lemma 6.1 6.3, respectively.

Proof. Let $\Phi(x, \lambda) = M(\vec{x}, \lambda) e^{-(\lambda X + \frac{1}{\lambda} \sigma_1(X))}$. Then

$$\begin{aligned} \frac{\partial \Phi}{\partial x} &= \sum_{j=1}^n w_j \partial_{x_j} \Phi \\ &= \sum_{j=1}^n w_j (-\lambda a_j - \frac{1}{\lambda} B_j - C_j) \Phi \end{aligned}$$

by (6.4). Therefore, using formula (6.19),

$$\begin{aligned}\frac{\partial \Psi}{\partial x} &= (-\lambda b(\sum_{j=1}^n w_j a_j) b^{-1} - \frac{1}{\lambda} b(\sum_{j=1}^n w_j B_j) b^{-1} + \sum w_j (-b C_j b^{-1} + (\partial_j b) b^{-1})) \Psi \\ &= (-\lambda b a b^{-1} - \frac{1}{\lambda} b(\sum_{j=1}^n w_j B_j) b^{-1} - v) \Psi.\end{aligned}\tag{6.21}$$

with $v = \sum w_j (b C_j b^{-1} - (\partial_j b) b^{-1}) \in \mathcal{S}_0$ by Lemma 6.3. The proof reduces to showing the $\sigma_1(b(\sum_{j=1}^n w_j B_j) b^{-1}) = b a b^{-1}$. Define

$$A = -\sum_{j=1}^n w_j b a_j b^{-1} dx_j,\tag{6.22}$$

$$B = -\sum_{j=1}^n w_j b B_j b^{-1} dx_j,\tag{6.23}$$

$$C = \sum_{j=1}^n w_j (-b C_j b^{-1} + (\partial_j b) b^{-1}) dx_j\tag{6.24}$$

and write (6.21) as

$$d\Psi = \lambda A \Psi + \frac{1}{\lambda} B \Psi + C \Psi.$$

So $d^2 \Psi = 0$. This implies

$$\begin{aligned}dA + A \wedge C + C \wedge A &= 0, \\ dB + B \wedge C + C \wedge B &= 0.\end{aligned}$$

Thus

$$d(A - \sigma_1(B)) + (A - \sigma_1(B)) \wedge C + C \wedge (A - \sigma_1(B)) = 0\tag{6.25}$$

by (6.20), (6.24). Along the direction $x(w_1, \dots, w_n)$, as $x \rightarrow -\infty$, we obtain

$$\begin{aligned}& A - \sigma_1(B) \\ &= \sum_{j=1}^n -w_j b a_j b^{-1} dx_j + w_j \sigma_1(b B_j b^{-1}) dx_j \\ &= \sum_{j=1}^n w_j b (-a_j + b^{-1} \sigma_1(b B_j b^{-1}) b) b^{-1} dx_j \\ &\rightarrow 0\end{aligned}\tag{6.26}$$

by (5.1), (5.46), (6.11), (6.18). Consequently, (6.25), (6.26) yield

$$A = \sigma_1(B).$$

□

Remark 5. For $i \in \{0, 1, \dots, n-1\}$, replacing σ_1 by σ_i defined in Remark 1, we can solve the associated inverse problem by analogy.

7 The Cauchy problem

Theorem 7.1. *Let $a, \tilde{a} \in \mathcal{A}$ be constant oblique directions, $b_0(x) \in K'_0$, $v_0(x) \in \mathcal{S}_0$. Suppose $b_0 - 1, v_0$ and their derivatives are rapidly decreasing as $|x| \rightarrow \infty$ and the set Z in Theorem 4.1 is a finite set contained in $\mathbb{C} \setminus \Sigma_a$. In case of $q \neq 0$, (b_0, v_0) satisfies additionally the small data constraint $|b_0 - 1|_{L^1_1} + |v_0|_{L^1_1} \ll 1$. Then the Cauchy problem of the twisted flow*

$$\left[\partial_x + bab^{-1}\lambda + v + \sigma_1(bab^{-1})\lambda^{-1}, \partial_t + \sum_{s=1}^{2j+1} Q_s \lambda^s + Q_0 + \sum_{s=1}^{2j+1} \sigma_1(Q_s) \lambda^{-s} \right] = 0,$$

$$b(x, 0) = b_0, \quad v(x, 0) = v_0$$

admits a smooth solution for $x \in \mathbb{R}, t \geq 0$. Here $Q_s(x, t)$, $0 \leq s \leq 2j + 1$, are defined by (2.11), Lemma 2.3.

Proof. We first apply Theorem 4.1, Corollary 4.1, Definition 4.4 and Theorem 4.2 to obtain the scattering data $(U_{\nu,0}^\pm(\lambda), V_0(\lambda))$ for the potential (b_0, v_0) . Define

$$U_\nu^\pm(\lambda, t) = e^{-t(\lambda^{2j+1}\tilde{a} + \frac{1}{\lambda^{2j+1}}\sigma_1(\tilde{a}))} U_{\nu,0}^\pm(\lambda) e^{t(\lambda^{2j+1}\tilde{a} + \frac{1}{\lambda^{2j+1}}\sigma_1(\tilde{a}))}, \quad \lambda \in \Omega_\nu^\pm \quad (7.1)$$

$$V(\lambda, t) = e^{-t(\lambda^{2j+1}\tilde{a} + \frac{1}{\lambda^{2j+1}}\sigma_1(\tilde{a}))} V_0(\lambda) e^{t(\lambda^{2j+1}\tilde{a} + \frac{1}{\lambda^{2j+1}}\sigma_1(\tilde{a}))}, \quad \lambda \in \Sigma_a. \quad (7.2)$$

Hence (U_ν^\pm, V) satisfies (4.27)-(4.37) and $|V - 1|_{L^\infty} \ll 1$ if $q \neq 0$ [4]. Thus one can apply Theorem 5.1, 5.2, and 6.1 to construct $M(x, t, \lambda)$, $b(x, t)$, $v(x, t)$. Let

$$m(x, t, \lambda) = b(x, t)M(x, t, \lambda),$$

$$\Psi(x, t, \lambda) = m(x, t, \lambda) e^{-x(\lambda a + \frac{1}{\lambda}\sigma_1(a)) - t(\lambda^{2j+1}\tilde{a} + \frac{1}{\lambda^{2j+1}}\sigma_1(\tilde{a}))}.$$

Then

$$\frac{\partial \Psi}{\partial x} = -\lambda bab^{-1}\Psi - \frac{1}{\lambda}\sigma_1(bab^{-1})\Psi - v\Psi,$$

$$b \in K'_0 \cap C^\infty, \quad v \in \mathcal{S}_0 \cap C^\infty.$$

So $\Psi(x, t, \lambda) \in L_+^{\sigma_0}$ and $\frac{\partial \Psi}{\partial t} \Psi(x)^{-1} \in \mathcal{L}_+^{\sigma_0}$. Moreover,

$$\begin{aligned} & \frac{\partial \Psi}{\partial t} \Psi(x)^{-1} \\ = & \left\{ \frac{\partial}{\partial t} \left(m(x, t, \lambda) e^{-x(\lambda a + \frac{1}{\lambda}\sigma_1(a)) - t(\lambda^{2j+1}\tilde{a} + \frac{1}{\lambda^{2j+1}}\sigma_1(\tilde{a}))} \right) \right\} \Psi^{-1} \\ = & \left\{ \left[\frac{\partial m}{\partial t} - m(\lambda^{2j+1}\tilde{a} + \frac{1}{\lambda^{2j+1}}\sigma_1(\tilde{a})) \right] e^{-x(\lambda a + \frac{1}{\lambda}\sigma_1(a)) - t(\lambda^{2j+1}\tilde{a} + \frac{1}{\lambda^{2j+1}}\sigma_1(\tilde{a}))} \right\} \\ & \cdot e^{x(\lambda a + \frac{1}{\lambda}\sigma_1(a)) + t(\lambda^{2j+1}\tilde{a} + \frac{1}{\lambda^{2j+1}}\sigma_1(\tilde{a}))} m^{-1} \\ = & \frac{\partial m}{\partial t} m^{-1} - m \left(\lambda^{2j+1}\tilde{a} + \frac{1}{\lambda^{2j+1}}\sigma_1(\tilde{a}) \right) m^{-1} \\ = & -\hat{\pi}_+(m J_{\tilde{a}, 2j+1} m^{-1}) \\ = & - \left(\sum_{s=1}^{2j+1} Q_s \lambda^s + Q_0 + \sum_{s=1}^{2j+1} \sigma_1(Q_s) \lambda^{-s} \right) \end{aligned}$$

by (2.11), Lemma 2.1-2.3. □

Theorem 7.2. *Let $a = \text{diag}(w_1, \dots, w_n) \in \mathcal{A}$ be a constant oblique direction, $b_0(x) \in K'_0$, $v_0(x) \in \mathcal{S}_0$, $b_0 - 1$, v_0 and their derivatives are rapidly decreasing as $|x| \rightarrow \infty$ and the set Z in Theorem 4.1 is a finite set contained in $\mathbb{C} \setminus \Sigma_a$. In case of $q \neq 0$, (b_0, v_0) satisfies additionally the small data constraint $|b_0 - 1|_{L^1_1} + |v_0|_{L^1_1} \ll 1$. Then there exists a solution to the 1-dimensional twisted $\frac{O(J, J)}{O(J) \times O(J)}$ -system (3.1) satisfying*

$$b(x(w_1, \dots, w_n)) = b_0(x), \quad \sum_{k=1}^n v_k(x(w_1, \dots, w_n)) = v_0(x).$$

Proof. We first apply Theorem 4.1, Corollary 4.1, and Theorem 4.2 to obtain the scattering data $(U_{\nu,0}^\pm(\lambda), V_0(\lambda))$ for the potential (b_0, v_0) . Then we apply Theorem 5.1, 5.2, 6.1 to construct $C_k(x_1, \dots, x_n)$, $b(x_1, \dots, x_n)$, $v_k(x_1, \dots, x_n)$, $M(x_1, \dots, x_n)$, such that $\Psi(x_1, \dots, x_n) = b(x_1, \dots, x_n)M(x_1, \dots, x_n)e^{-\lambda \sum x_k a_k - \frac{1}{\lambda} \sum \sigma_1(x_k a_k)}$ satisfies

$$\frac{\partial \Psi}{\partial x_k} = -\lambda b a_k b^{-1} \Psi - v_k \Psi - \frac{1}{\lambda} \sigma_1(b a_k b^{-1}) \Psi,$$

with a_k defined by (3.2), $v_k = b C_k b^{-1} - (\partial_k b) b^{-1}$, and

$$\begin{aligned} b_0(x) &= b(x(w_1, \dots, w_n)) \in K'_0 \cap C^\infty, \\ v_0(x) &= \sum_{k=1}^n v_k(x(w_1, \dots, w_n)) \in \mathcal{S}_0 \cap C^\infty. \end{aligned}$$

□

Acknowledgements

The first author was partially supported by NSFC grant No. 10971111 and the second author was partially supported by NSC 99-2115-M-001-002. The second author would like to thank Professor C. L. Terng for introducing twisted hierarchies, Professor V. Sokolov for providing important references, and Professor M. Guest for his interest and encouragement.

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